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Journal of Geometry and Physics 29 (1999) 87–150

JOURNAL OF GEOMETRY AND PHYSICS

Smooth coalgebras

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Received 9 April 1998

Abstract

A complete mathematical framework for coalgebraic formulation of supergeometry and its infinite-dimensional extension is proposed. Within this approach a supermanifold is defined as a graded coalgebra endowed with a smooth structure. The category of such coalgebras is constructed and analysed. It is shown that it contains as its full subcategories both the category of smooth Fréchet manifolds and the category of finite-dimensional Berezin–Leites–Kostant supermanifolds. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Noncommutative geometry; Quantum groups
1991 MSC: 58A50; 16W30; 58B99
Keywords: Graded manifolds; Infinte-dimensional manifolds; Dual coalgebras; Graded hopf algebras

1. Introduction

There are basically two different approaches to supergeometry: the algebraic approach introduced by Berezin and Leites [8] and further developed by Kostant [19] and Leites [22], cf. [7,12,25,35]; and the geometrical approach proposed by Rogers [31] and DeWitt [11], cf. [2,3,9,18,30]. The theoretical framework incorporating both approaches was first proposed by Rothstein in the form of axiomatic definition of a supermanifold [33] and further analysed and improved by Bruzzo et al. [4]. The Berezin–Leites–Kostant (BLK) theory provides the simplest realisation of this axiomatic definition perfectly sufficient to derive all nontrivial results of finite-dimensional supergeometry including theories of: Lie supergroups [19], complex supermanifolds [15,29,32], supersymplectic supermanifolds [13,34], or moduli of super Riemann surfaces [1,6,14,21].

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PII S0393-0440(98)00033-3

All these results supported by methods of algebraic and analytic geometry along with conceptual simplicity of the BLK approach (it does not contain any spurious Grassman algebra of constants) make it a perfect mathematical language for all physical applications in which a finite-dimensional geometry is involved. In supersymmetric classical and quantum field theories however one has to deal with infinite-dimensional superspaces of supergeometric stcrutures. A typical problem one encounters in this type of applications is to analyse the global structure of supermoduli intuitively constructed as a quotient of an infinite-dimensional supermanifold of field configurations by an infinite-dimensional supergroup of gauge transformations. This construction well known for "bosonic" models was never made rigorous in the super case. The only method available is based on finite-dimensional techniques of deformation theory [1,14,21]. Although it usually provides quite a lot of information about global geometry of the supermoduli, analysing induced structures seems to require constructing the quotient. The lack of rigorous and efficient methods of infinite-dimensional supergeometry is also responsible for the informal heuristic way one treats anticommuting classical fields in physical models. As a consequence the understanding of global geometry of supermanifolds of field configurations as well as actions of gauge supergroups on these supermanifolds is in sharp contrast with sophisticated methods of standard global functional analysis [16,28] and detailed knowledge about similar problems in "bosonic" models.

The aim of the present paper is to construct an infinite-dimensional extension of the BLK supergeometry. Before discussing a possible solution to this problem, let us briefly consider what kind of examples of infinite-dimensional supermanifolds one should expect in physical models. In the standard smooth geometry the most important and interesting class of objects studied via methods of functional nonlinear analyses are manifolds of maps with possibly additional properties like that carried by sections of bundles. In particular manifolds of various geometrical structures belong to this class which is actually essential for the physical and most of mathematical applications of infinite-dimensional geometry [16,28]. One can expect that also in supergeometry, supermanifolds of maps are fundamental for a geometric formulation of supersymmetric models. For an excellent heuristic discussion of the notion of map between supermanifolds in the context of physical applications we refer to the paper by Nelson [27]. A special case of maps from the supermanifold $S^{1,1}$ to a manifold was also analysed by Lott [23,24]. The main points is that supergeometry requires *a notion of map essentially wider than the notion of morphism.* This is in contrast to the standard smooth geometry where both notions coincide.

In the BLK category, morphisms are defined as even $\mathbb{Z}_2$-graded algebra morphisms. For instance for any pair $\mathcal{A}, \mathcal{B}$ of supermanifolds all BLK morphisms from $\mathcal{A}$ to $\mathcal{B}$ form an ordinary (not graded) infinite-dimensional manifold $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$. One would rather expect a supermanifold of maps $\mathrm{Map}(\mathcal{A}, \mathcal{B})$ with $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$ playing the role of its underlying manifold. In particular one would like to interpret the $\mathbb{Z}_2$-graded space of real-valued superfunctions on a finite-dimensional supermanifold $\mathcal{A}$ as a model space of a linear infinite-dimensional supermanifold of maps from $\mathcal{A}$ to $\mathbb{R}$. Certainly morphisms from $\mathcal{A}$ to $\mathbb{R}$ form only the even part of this superspace.

The fact that morphisms are not enough to capture the intuitive notion of "odd" maps one needs in physical application is sometimes referred to as the main shortcoming of the BLK theory (see for example the discussion in [10]). In its simplest form the argument says that coefficients of a superfunction are ordinary real-valued functions which do not anticommute and therefore cannot provide a working model for anticommuting classical fields one needs in physics.

This apparent drawback can be simply overcome by regarding anticommuting classical fields as odd coordinates on an infinite-dimensional supermanifold of fields configurations [36]. Within the BLK approach the odd variables $\{\theta_\alpha\}_{\alpha=1}^n$ can be seen as a basis in the odd part of a $\mathbb{Z}_2$-graded model space $\mathbb{R}^{m,n} = \mathbb{R}^m \oplus \mathbb{R}^n$. As such they are genuine "commuting" objects of standard linear algebra. The "anticommuting" nature shows up when elements of the dual basis $\{\theta^\alpha\}_{\alpha=1}^n$ are interpreted as generators of the exterior algebra $\wedge(\mathbb{R}^n)'$ over $[\mathbb{R}^n]'$. Following this line of thinking one can regard classical fermion fields as elements of the ordinary linear space $\mathcal{F}_1$ of sections of an appropriate bundle, with $\mathcal{F}_1$ being the odd part of an infinite-dimensional $\mathbb{Z}_2$-graded model space $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$. Elements of $\mathcal{F}_1$ anticommute as arguments of functionals from the exterior algebra $\wedge\mathcal{F}_1'$ and play essentially the same role as θ-variables in the finite-dimensional case.

It should be stressed that a reasonable extension of an ordinary manifold $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$ to supermanifold $\mathrm{Map}(\mathcal{A}, \mathcal{B})$ requires a global constuction. Indeed according to the basic idea of the BKL approach one can think of "odd maps" as odd coordinates of an infinite-dimensional supermanifolds of maps $\mathrm{Map}(\mathcal{A}, \mathcal{B})$ rather than elements of some set. This means in particular that also the notion of composition cannot be defined point by point but rather as a morphism of supermanifolds

$$\circ : \mathrm{Map}(\mathcal{A}, \mathcal{B}) \times \mathrm{Map}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Map}(\mathcal{A}, \mathcal{C}),$$

where $\times$ stands for the direct product in the category of infinite-dimensional supermanifolds. The obvious requirement for composition $\circ$ is that its underlying map coincides with the standard composition of morphisms of finite-dimensional supermanifolds.

Another problem of constructing an infinite-dimensional supergeometry is to choose an appropriate class of model spaces. Since the composition of morphisms in the BLK category involves differentiation of their coefficient functions with respect to even coordinates, smoothness is the minimal possible requirement for morphisms and functions. In consequence supermanifolds of supergeometrical structures which are expected to be most interesting objects of infinite-dimensional supergeometry should be modelled on Fréchet spaces.

The first systematic formulation of infinite-dimensional supergeometry was given by Molotkov [26]. In this approach Banach supermanifolds are defined as functors from the category of finite-dimensional real Grassmann superalgebras $\wedge\mathbb{R}^n$ ($n = 1, 2, \ldots$) to the category of smooth Banach manifolds

$$\mathcal{M} : \wedge\mathbb{R}^n \longrightarrow \mathcal{M}(\wedge\mathbb{R}^n).$$

For each Grassman algebra $\wedge\mathbb{R}^n$, $\mathcal{M}(\wedge\mathbb{R}^n)$ can be identified with smooth manifold of morphisms $\mathrm{Mor}(\mathcal{P}_n, \mathcal{M})$ where $\mathcal{P}_n$ denotes finite-dimensional supermanifold with zero even dimension and the odd dimension n (n-dimensional superpoint).

The idea to regard supermanifolds as point functors was first introduced by Schwarz in his attempt to reconcile the standard sheaf formulation of BLK finite-dimensional geometry with the intuitive informal language used by physicists [39]. The equivalence of Schwarz's approach with the BLK theory was shown by Voronov in [41]. Molotkov's formulation can be seen as a proper generalisation of the Schwarz description to infinite-dimensions (i.e. the BLK category of finite-dimensional supermanifolds is a full subcategory of the category of smooth Banach supermanifolds).

For any two supermanifolds $\mathcal{A}$, $\mathcal{B}$ (not necessarily finite-dimensional) the supermanifold of maps can be defined by the functor

$$\mathrm{Map}(\mathcal{A}, \mathcal{B}) : \wedge\mathbb{R}^n \longrightarrow \mathrm{Mor}(\mathcal{P}_n \times \mathcal{A}, \mathcal{B}).$$

The formalism also allows for a construction of a composition with the required properties and applies as well to smooth supermanifolds modelled on locally convex or tame Fréchet superspaces. In principle, Molotkov's formulation satisfies all requirements a mathematically rigorous infinite-dimensional supergeometry should satisfy. It has been in fact implicitly used in several papers when a rigorous treatment of elements of infinite-dimensonal supergeometry was unavoidable [1,21,23,24]. However, technical and conceptual difficulties of this approach make its wider application in physics highly problematic. According to the basic idea of the functorial approach, an object $\mathcal{A}$ in the category is fully described by morphisms $\mathrm{Mor}(\mathcal{P}_n, \mathcal{A})$ from a sufficiently large family $\{\mathcal{P}_n\}_{n\in I}$ of other objects. Such a description is in sharp contrast with the intuitive physical understanding of space or superspace. Also technicalities involved are in contrast with relatively simple heuristic formalism used by physicists.

Another approach to infinite-dimensional supergeometry aimed to avoid the functorial definition of supermanifolds was developed by Schmitt [37,38]. The basic idea is to define an infinite-dimensional supermanifold as a ringed space. Although not functorial, this approach is technically even more complicated. The main reason is that in the infinite-dimensional supergeometry the language and methods of algebraic geometry are essentially less efficient and less powerful than in the finite-dimensional case. In the standard BLK approach one has a very simple algebraic description of morphisms between supermanifolds, either as morphisms of sheaves of graded algebras or as morphisms of graded algebras of functions. Also vector fields on a supermanifold can be described in a purely algebraic way as graded derivations of the graded algebra of superfunctions. Proceeding to infinite-dimensional geometry one can still consider sheaves of smooth functionals but the simple algebraic descriptions of morphisms and vector fields are no longer available. Additional conditions involving topology as well as differential calculus on the infinite-deimensional model spaces are required in both cases. In fact technical difficulties involved were overcome only in the case of real-analytic and complex-analytic supermanifolds [37,38] which essentially restricts possible physical applications of the theory. It is also not clear how to construct supermanifolds of maps and composition within this approach.

The formalisms of both approaches seem to be technically too difficult when compared with relatively simple heuristic rules used by physicists even in most complicated geometrical supersymmetric models. This suggests that there might be a simpler theory incorporating all desired features of the hitherto formulations but better suited for constructing and analysing examples arising in physical applications.

The aim of the present paper is to construct an alternative coalgebraic formulation of infinite-dimensional supergeometry which allows to avoid at least some of the technicalities of the functorial and the sheaf descriptions. The idea of such an approach was first proposed in the excellent paper by Batchelor [5] where a candidate for the dual coalgebra of the supermanifold of maps between finite-dimensional supermanifolds was constructed and analysed.

For any associative algebra with unit (A, m, u), let us denote by A° the largest subspace of the full algebraic dual A' such that $m'(A^\circ) \subset A^\circ \otimes A^\circ$, where $m' : A' \to (A \otimes A)'$ is the map dual to the multiplication $m : A \otimes A \to A$. A° with comultiplication given by m' and counit given by u' is called the dual coalgebra of A. In the case of the algebra $C^\infty(M)$ of smooth functions on a finite-dimensional manifold M, $C^\infty(M)^\circ$ is called the dual coalgebra of M and consists of all finite linear combinations of Dirac delta functions and their partial derivatives. In the context of finite-dimensional supergeometry dual ($\mathbb{Z}_2$-graded) coalgebras were first analysed and extensively used by Kostant in his theory of Lie supergroups [19].

The idea of Batchelor's approach is to consider the dual algebra of a supermanifold as a fundamental object. The crucial notion introduced in [5] is that of mapping coalgebra $P(\mathcal{A}, \mathcal{B})$ defined for any two finite-dimensional supermanifolds $\mathcal{A}$, $\mathcal{B}$ in terms of universal coalgebra measuring the algebra of superfunctions on $\mathcal{B}$ to the algebra of superfunctions on $\mathcal{A}$. Although the full structure theorem for $P(\mathcal{A}, \mathcal{B})$ has not been proven, the mapping coalgebra has many expected properties of the dual coalgebra of the supermanifold of maps $\mathrm{Map}(\mathcal{A}, \mathcal{B})$. In particular, the space of group-like elements of $P(\mathcal{A}, \mathcal{B})$ coincides with $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$. Moreover, there exists a simple definition of composition which leads to the expected Hopf algebra structure in the case of superdiffeomorphisms. Batchelor's construction can also be extended to coalgebras corresponding to supermanifolds of sections.

In the original paper [5], only the algebraic structure of mapping coalgebra has been analysed. This is certainly not enough to define supermanifold in terms of its dual coalgebra. A pure coalgebra structure has to be supplemented by analytic data encoding a smooth structure on a supermanifold. These additional data are also necessary to select those coalgebra morphisms which correspond to smooth morphisms of supermanifolds. Extra conditions in the definition of morphisms may seem to be a shortcoming of the coalgebraic approach in comparison to the algebraic one where smooth morphisms can be defined in a purely algebraic way. Let us however recall that the simple algebraic definition does not work in the case of infinite-dimensional model spaces. Moreover, the detailed discussion of morphisms within Schmitt's sheaf formulation of infinite-dimensional supergeometry shows that the coalgebraic structure is essential for an appropriate definition of smoothness or analyticity [37]. On the other hand (as we shall see in the following) the extra conditions one has to impose on coalgebraic maps are essentially identical to the differentiability condition one imposes on maps of sets in the traditional definition of smooth morphisms between manifolds.

In the present paper we propose an intrinsic way to handle the additional analytic data necessary to describe smooth structures. The main result is the construction of the category of smooth coalgebras which contains as its full subcategories both the BLK category of finite-dimensional supermanifolds and the category of smooth Fréchet manifolds. This provides a complete theoretical framework of the coalgebraic formulation of supergeometry. Although Batchelor's results [5] were our main motivation, we leave the construction of smooth structure on the mapping coalgebra $P(\mathcal{A}, \mathcal{B})$ for future publications. This involves in particular a construction of smooth atlas on the manifold $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$, which goes far beyond the scope of this paper.

It should be stressed that the coalgebraic description of supermanifolds has its advantages even in the finite dimensions. First of all the general structure of the theory is similar to that of the standard smooth geometry: supermanifolds are defined as sets with extra structure and morphisms as maps of sets (with arrows in the "right" direction) preserving these structures. Secondly the direct product in the category is just the algebraic tensor product of coalgebras which makes many of the standard geometric constructions much simpler and more intuitive than in the sheaf or the functorial approaches. Finally the coalgebraic techniques proved to the very useful in Kostant's theory of Lie supergroups [19]. In fact this theory gets much simpler when smooth coalgebra morphisms are defined in the intrinsic coalgebraic language without referring to the algebraic formulation.

The content of the paper is as follows. Section 2 contains preliminary material necessary for further constructions. In Section 2.1 the basic facts about symmetric tensor algebra $S(X)$ of $\mathbb{Z}_2$-graded vector space $X = X_0 \oplus X_1$ are presented. In particular, Hopf algebra structure on $S(X)$ is described and a less known universal property of $S(X)$ with respect to its coalgebraic structure is proven. This property is crucial for our description of smooth coalgebraic maps. In Sections 2.2 and 2.3 we recall some properties of the model category **fm** of Fréchet manifolds and the model category **sm** of BLK supermanifolds, respectively. In Section 2.4 the category of BLK finite-dimensional supermanifolds is briefly presented. This well-known material is included for notational purposes as well as for providing some motivation for further constructions.

In Section 3 model category **sc** of smooth coalgebras is defined. In Section 3.1 we introduce open coalgebras as objects of the model category. In Section 3.2 we present a crucial (for all coalgebraic formulation) notion of smooth coalgebra morphism and prove that it satisfies all the required properties. In particular, the component description of morphisms is introduced and the formula for the composition is derived. In Section 3.3 the construction of model category **sc** is completed and the direct product in **sc** is analysed. Finally, in Section 3.4, we prove that the model category **fm** of Fréchet manifolds can be identified with the full subcategory $\mathbf{sc}_0$ of even open coalgebras, and the model category **sm** of BLK supermanifolds can be identified with the full subcategory $\mathbf{sc}^{<}$ of finite-dimensional open coalgebras. This shows that **sc** is an appropriate extension of **fm** and **sm**. In Section 3.5 the notion of superfunction on an open coalgebra is introduced and analysed.

In Section 4 we describe construction and main properties of the category **SC** of smooth coalgebras. In Section 4.1 the smooth coalgebra is defined as a collection of objects from the model category **sc** glued together with a collection of compatible morphisms from **sc**. Smooth morphisms of smooth coalgebras are defined along standard lines by requiring that their local expressions are morphisms from **sc**. The notion of superfunction on a smooth coalgebra is defined and the functor from the category of smooth coalgebras **SC** to the category of sheaves of $\mathbb{Z}_2$-graded algebras is constructed. In Section 4.2 the direct product in the category **SC** is analysed. In Section 4.3 we prove that the full subcategory $\mathbf{SC}_0$ of even smooth coalgebras is isomorphic with the category of Fréchet manifolds. In Section 4.4 the corresponding result for full subcategory $\mathbf{SC}^<$ of finite-dimensional smooth coalgebras and the category of BLK supermanifolds is derived.

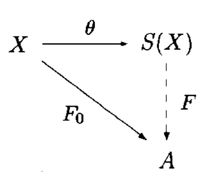
Appendix A contains definitions and basic facts about $\mathbb{Z}_2$-graded spaces (Appendix A.1), algebras (Appendix A.2), coalgebras (Appendix A.3), and bialgebras (Appendix A.4). Also some elementary material on dual coalgebras of finite-dimensional supermanifolds is briefly presented (Appendix A.5).

2. Preliminaries

2.1. Symmetric algebra of graded vector space

Definition 2.1.1. Let $X = X_0 \oplus X_1$ be a $\mathbb{Z}_2$-graded space. A symmetric algebra of X is a pair $(S(X), \theta)$ where $S(X)$ is a $\mathbb{Z}_2$-graded commutative algebra and $\theta : X \to S(X)$ a morphism of $\mathbb{Z}_2$-graded space such that the following universal property is satisfied.

For every $\mathbb{Z}_2$-graded commutative algebra A and every morphism $F_0 : X \to A$ of $\mathbb{Z}_2$-graded spaces there exists a unique $\mathbb{Z}_2$-graded algebra morphism $F : S(X) \to A$ making the diagram



commute.

The uniqueness of $S(X)$ is a standard consequence of the universal property. The existence can be shown by explicit construction of $S(X)$ as the quotient algebra

$$\Pi : T(X) \longrightarrow T(X)/I(X) \equiv S(X),$$

where $(T(X), \theta_T : X \to T(X))$ is the tensor algebra of X and $I(X)$ is the ideal generated by elements

$$a \otimes b - (-1)^{|a||b|} b \otimes a,$$

where $a, b \in X_0 \cup X_1$, and $|\cdot|$ denotes the parity of an element. The ideal $I(X)$ is homogeneous with respect to the canonical $\mathbb{Z}_2 \oplus \mathbb{Z}_+$ bigrading on $T(X)$ and $S(X)$ acquires the structure of bigraded commutative algebra

$$S(X) = \bigoplus_{\substack{k \geq 0 \\ i=0,1}} S_i^k(X),$$

where $S^k(X) = S_0^k(X) \oplus S_1^k(X)$ is the kth symmetric tensor power of X. Since $I(X)$ is generated by elements of degree 2, one has the identifications

$$S^0(X) = T^0(X) = \mathbb{R}, \qquad S^1(X) = T^1(X) = X,$$

and the canonical map $\theta = \Phi \circ \theta_T : X \to S(X)$ is injective.

The algebra $S(X)$ is generated by the set $\{1\} \cup X_0 \cup X_1$, i.e. every element $\omega \in S(X)$ can be represented as a finite sum of monomials of homogeneous elements of X and 1.

Proposition 2.1.1. *Let X, Y be $\mathbb{Z}_2$-graded spaces and $\theta_X : X \to S(X)$, $\theta_Y : Y \to S(Y)$ the canonical inclusions into the corresponding symmetric algebras. Then the universal extension*

$$\kappa : S(X \oplus Y) \longrightarrow S(X) \otimes S(Y)$$

of the map

$$\kappa_0 : X \oplus Y \ni (a, b) \longrightarrow \theta_X a \otimes 1 + 1 \otimes \theta_Y b \in S(X) \otimes S(Y)$$

is an isomorphism of bigraded algebras.

Remark 2.1.1. By Proposition 2.1.1 for each $\mathbb{Z}_2$-graded space $X = X_0 \oplus X_1$ there is the canonical isomorphism of bigraded algebras

$$\bar{\kappa} : S(X_0 \oplus X_1) \longrightarrow S(X_0) \otimes \wedge(X_1),$$

where $S(X_0)$ is the usual symmetric algebra with its canonical $\mathbb{Z}_+$-grading and the trivial $\mathbb{Z}_2$-grading $(S(X_0))_1 = \{0\})$, and $\wedge(Y_1)$ is the usual exterior algebra of the vector space Y_1 with its canonical $\mathbb{Z}_2 \oplus \mathbb{Z}_+$ bigrading.

Proposition 2.1.2. *Let X be a $\mathbb{Z}_2$-graded space and $S(X)$ its symmetric algebra. Let $\Delta : S(X) \to S(X) \otimes S(X)$ be the universal extention of the map*

$$d : X \ni a \longrightarrow a \otimes 1 + 1 \otimes a \in S(X) \otimes S(X),$$

and $\varepsilon : S(X) \longrightarrow \mathbb{R}$ the universal extention of the map $0 : X \ni a \to 0 \in \mathbb{R}$.

Then $(S(X), \Delta, \varepsilon)$ is a commutative cocommutative $\mathbb{Z}_2$-graded bialgebra.

Remark 2.1.2. We shall introduce some notational conventions which will be used in various contexts in the following.

A k-partition of the index set $\{1, \ldots, n\}$ is defined as a sequence $\mathcal{P} = \{P_1, \ldots, P_k\}$ of disjoint (possibly empty) subsets of the index set such that $\{1, \ldots, n\} = P_1 \cup \cdots \cup P_k$. A k-partition is *nonempty* if $P_i \neq \emptyset$ for all $i = 1, \ldots, k$. Note that a nonempty n-partition of the index set $\{1, \ldots, n\}$ is a permutation of $\{1, \ldots, n\}$.

Let $\mathcal{X} = \{a_i\}_{i=1}^n$ be a sequence of nonvanishing homogeneous elements of a graded space X. For every nonempty subset P of the index set $\{1, \ldots, n\}$ we define

$$a_P = a_{p_1} \cdots a_{p_l} \in S^l(X),$$

where $\{p_1, \ldots, p_l\} = P$ and $p_1 \leq \cdots \leq p_l$. We denote the number of elements of P by $|P|$. If P is empty we set $a_P = 1$ and $|P| = 0$.

For every k-partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of the index set $\{1, \ldots, n\}$ we define the number $\sigma(\mathcal{X}, \mathcal{P}) = \pm 1$ uniquely determined by the relation

$$a_1 \cdots a_k = \sigma(\mathcal{X}, \mathcal{P}) a_{P_1} \cdots a_{P_k}.$$

Proposition 2.1.3. *Let $(S(X), \Delta, \varepsilon)$ be the coalgebra of Proposition* 2.1.2, *and $\mathcal{X} = \{a_i\}_{i=1}^n$ be a sequence of nonvanishing homogeneous elements of a graded space X. Then:*
1. $\Delta(1) = 1 \otimes 1$, *and* $\varepsilon(1) = 1$.
2. *For every $k \geq 1$*

$$\Delta^k(a_1 \cdots a_n) = \sum_{\mathcal{P}=\{P_1,\ldots,P_{k+1}\}} \sigma(\mathcal{X}, \mathcal{P}) a_{P_1} \otimes \cdots \otimes a_{P_{k+1}} \tag{1}$$

where the sum runs over all $(k+1)$-partitions of the index set $\{1, \ldots, n\}$.

3. $\varepsilon(a_1 \cdots a_n) = 0$.

Proposition 2.1.4. *The symmetric algebra $S(X)$ of a $\mathbb{Z}_2$-graded space X with the coalgebraic structure of Proposition* 2.1.2 *is a strictly bigraded cocommutative coalgebra.*

Remark 2.1.3. It follows from Proposition 2.1.4 that $S(X)$ is a pointed irreducible coalgebra. The relation between the $\mathbb{Z}_+$-grading $S(X) = \oplus_{i=0}^{\infty} S^i(X)$ and the coradical filtration $S(X) = \cup_{k \geq 0} S^{(k)}(X)$ is given by

$$S^{(k)}(X) = \bigoplus_{i=0}^{k} S^i(X).$$

The coalgebraic structure of $S(X)$ introduced in Proposition 2.1.2 is universal in the following sense.

Theorem 2.1.1. *Let $S(X)$ be the symmetric algebra of a $\mathbb{Z}_2$-graded vector space X. There exists a unique extension of the bigraded commutative algebra structure on $S(X)$ to a strictly bigraded commutative cocommutative Hopf algebra structure on $S(X)$.*

Remark 2.1.4. The antipode $s : S(X) \to S(X)$ is given by the universal extension of the map

$$- : X \ni a \longrightarrow -a \in S(X)_{\mathrm{op}},$$

where $S(X)_{op}$ is the bigraded space $S(X)$ with the "opposite" algebra structure given by $M_{op}(a \otimes b) = (-1)^{|a||b|} M(b \otimes a)$, $u_{op} = u$. One can easily show that for arbitrary homogeneous elements $a_1, \ldots, a_n \in X$, $s(a_1 \cdots a_n) = (-1)^n a_n \cdots a_1$.

Definition 2.1.2. Let $(\mathcal{C}, \Delta, \varepsilon)$ be a $\mathbb{Z}_2$-graded coalgebra, $(\mathcal{A}, M, u)$ a $\mathbb{Z}_2$-graded algebra, and $\mathrm{Hom}(\mathcal{C}, \mathcal{A})$ the space of linear maps from $\mathcal{C}$ to $\mathcal{A}$. For any $f, g \in \mathrm{Hom}(\mathcal{C}, \mathcal{A})$ we define

$$f * g \equiv M \circ (f \otimes g) \circ \Delta.$$

By definition $f * g \in \mathrm{Hom}(\mathcal{C}, \mathcal{A})$ and $|f * g| = |f| + |g|$ (with respect to the standard $\mathbb{Z}_2$-grading in $\mathrm{Hom}(\mathcal{C}, \mathcal{A})$). One easily verifies that $\mathrm{Hom}(\mathcal{C}, \mathcal{A})$ with the multiplication $*$ and the identity $1_* \equiv u \circ \varepsilon$ is a $\mathbb{Z}_2$-graded algebra. The multiplication $*$ is called *convolution*.

The next theorem describes the universal property of $S(X)$ with respect to its coalgebra structure. This result is essential for our description of smooth coalgebra morphisms given in the Section 2.2.

Theorem 2.1.2. *Let $S(X)$ be the symmetric algebra of a $\mathbb{Z}_2$-graded space X and π^P: $S(X) \to S^1(X) = X$ the projection with respect to the $\mathbb{Z}_+$-grading in $S(X)$. Let $\mathcal{C}$ be a pointed irreducible $\mathbb{Z}_2$-graded cocommutative coalgebra and $\mathcal{C}^+$ the kernel of the counit $\varepsilon_{\mathcal{C}}$ in $\mathcal{C}$. Denote by $\rho^+ : \mathcal{C}^+ \to \mathcal{C}$ the inclusion and by $\pi^+ : \mathcal{C} \to \mathcal{C}^+$ the projection with respect to the direct sum decomposition $\mathcal{C} = \mathbb{R}\{p\} \oplus \mathcal{C}^+$, where p is the unique group-like element of $\mathcal{C}$. Then:*

1. *For every morphism $\Phi^+ : \mathcal{C}^+ \to X$ of $\mathbb{Z}_2$-graded spaces there exists a unique morphism $\Phi : \mathcal{C} \to S(X)$ of $\mathbb{Z}_2$-graded coalgebras such that the diagram*

$$\begin{array}{ccc} X & \xleftarrow{\ \pi^P\ } & S(X) \\ {\scriptstyle \Phi^+}\uparrow & & \uparrow{\scriptstyle \Phi} \\ \mathcal{C}^+ & \xrightarrow{\ \rho^+\ } & \mathcal{C} \end{array}$$

is commutative.

2. *The universal extension Φ is given by*

$$\Phi = * \exp \Phi^+ \equiv \sum_{k \geq 0} \frac{1}{k!} \Phi^{+k},$$

where

$$\Phi^{+0} \equiv u \circ \varepsilon_{\mathcal{C}},$$
$$\Phi^{+k} \equiv \underbrace{\Phi^+ \circ \pi^+ * \cdots * \Phi^+ \circ \pi^+}_{k}, \quad k \geq 1,$$

and $*$ is the convolution in $\mathrm{Hom}(\mathcal{C}, S(X))$.

Note that in the purely even case $X = X_0 \oplus \{0\}$, $S(X)$ with respect to its $\mathbb{Z}_+$-graded coalgebra structure is isomorphic with the universal pointed irreducible cocommutative coalgebra considered in [40]. Part 1 of the theorem above is a $\mathbb{Z}_2$-graded version of Theorem 12.2.5 in [40]. In the special case $\mathcal{C} = S(Y)$, where Y is a $\mathbb{Z}_2$-graded space, the explicit formula for the universal extension has been derived in [37]. The proof given here is a generalization of Schmitt's method.

Lemma 2.1.1. *With the notation of Theorem 2.1.2, for every $k \geq 0$ the following relation holds:*

$$\Delta \circ \Phi^{+k} = \sum_{i=0}^{k} \binom{k}{i} (\Phi^{+i} \otimes \Phi^{+k-i}) \circ \Delta_C.$$

Proof. The case $k = 0$ is straightforward. For $k = 1$ one has

$$\begin{aligned}\Delta \circ \Phi^{+1}(c) &= \Phi^{+1}(c) \otimes 1 + 1 \otimes \Phi^{+1}(c)\\ &= \sum_{(c)} (\Phi^{+1}(c_{(1)}) \otimes u \circ \varepsilon_C(c_{(2)}) + u \circ \varepsilon_C(c_{(1)}) \otimes \Phi^{+1}(c_{(2)}))\\ &= \sum_{i=1}^{1} \binom{1}{i} (\Phi^{+i} \otimes \Phi^{+1-i}) \circ \Delta_C(c).\end{aligned}$$

By definition of Φ^{+k}

$$\Phi^{+k} = M \circ (\Phi^{+k} \otimes \Phi^{+0}) \circ \Delta_C = M \circ (\Phi^{+0} \otimes \Phi^{+k}) \circ \Delta_C, \tag{2}$$

and

$$\Phi^{+k+1} = M \circ (\Phi^{+k} \otimes \Phi^{+1}) \circ \Delta_C = M \circ (\Phi^{+1} \otimes \Phi^{+k}) \circ \Delta_C, \tag{3}$$

for all $k \geq 1$. Then by the induction hypothesis and (3)

$$\begin{aligned}\Delta \circ \Phi^{+k+1} &= \Delta \circ M \circ (\Phi^{+k} \otimes \Phi^{+1}) \circ \Delta_C\\ &= (M \otimes M) \circ (\mathrm{id} \otimes T \otimes \mathrm{id}) \circ ((\Delta \circ \Phi^{+k}) \otimes (\Delta \circ \Phi^{+1})) \circ \Delta_C\\ &= (M \otimes M) \circ (\mathrm{id} \otimes T \otimes \mathrm{id})\\ &\quad \circ \sum_{i=0}^{k} \binom{n}{i} (\Phi^{+i} \otimes \Phi^{+k-i} \otimes (\Phi^{+1} \otimes \Phi^{+0} + \Phi^{+0} \otimes \Phi^{+1}))\\ &\quad \circ (\Delta_C \otimes \Delta_C) \circ \Delta_C\\ &= (M \otimes M)\\ &\quad \circ \sum_{i=0}^{k} \binom{n}{i} (\Phi^{+i} \otimes \Phi^{+1} \otimes \Phi^{+k-i} \otimes \Phi^{+0}\\ &\quad + \Phi^{+i} \otimes \Phi^{+0} \otimes \Phi^{+k-i} \otimes \Phi^{+1})\\ &\quad \circ (\mathrm{id} \otimes T \otimes \mathrm{id}) \circ (\Delta_C \otimes \Delta_C) \circ \Delta_C.\end{aligned}$$

As a consequence of the coassociativity and cocommutativity of Δ_C one has

$$(\mathrm{id} \otimes T \otimes \mathrm{id}) \circ (\Delta_C \otimes \Delta_C) \circ \Delta_C = (\Delta_C \otimes \Delta_C) \circ \Delta_C.$$

Using this relation and formulae (2) and (3) one finally gets

$$\begin{aligned}\Delta \circ \Phi^{+k+1} &= \sum_{i=0}^{k} \binom{n}{i} (\Phi^{+i+1} \otimes \Phi^{+k-1} + \Phi^{+i} \otimes \Phi^{+k+1-i}) \circ \Delta_C\\ &= \sum_{i=0}^{k+1} \binom{n+1}{i} (\Phi^{+i} \otimes \Phi^{+k+1-i}) \circ \Delta_C. \qquad \square\end{aligned}$$

Proof of Theorem 2.1.2. Existence. Let $\{\mathcal{C}^{(k)}\}_{k\geq 0}$ be the coradical filtration of $\mathcal{C}$. For any given $k \geq 0$ all terms $(1/l!)\Phi^{+l}$ with $l \geq k$ vanish on $\mathcal{C}^{(k)}$. By the structure theorem, Theorem A.3.2 (see Appendix A), for every $c \in \mathcal{C}$ there exists k such that $c \in \mathcal{C}^{(k)}$, hence $*\exp\Phi^+$ is well defined on $\mathcal{C}$. By construction $\Phi = *\exp\Phi^+$ is a morphism of graded spaces such that $\pi^P \circ \Phi \circ \rho^+ = \Phi^+$. Using Lemma 2.1.1 one has

$$\begin{aligned}\Delta \circ \Phi &= \sum_{k\geq 0} \frac{1}{k!}\binom{k}{i}(\Phi^{+i} \otimes \Phi^{+k-i}) \circ \Delta_C \\ &= \sum_{n\geq 0}\sum_{m\geq 0} \frac{1}{n!}\frac{1}{m!}(\Phi^{+n} \otimes \Phi^{+m}) \circ \Delta_C \\ &= (\Phi \otimes \Phi) \circ \Delta_C.\end{aligned}$$

For $k \geq 1$, $\Phi^{+k} \subset S(X)^+ = \ker\varepsilon$ and

$$\varepsilon \circ \Phi = \varepsilon \circ u \circ \varepsilon_C + \sum_{k\geq 0} \frac{1}{k!}\varepsilon \circ \Phi^{+k} = \varepsilon_C.$$

It follows that Φ is a morphism of graded algebras which completes the proof of existence.

Uniqueness. For any other extension Φ' we have $\pi^P \circ \Phi' = \pi^P \circ \Phi$ and therefore $\mathrm{Im}(\Phi' - \Phi) \cap P(S(X)) = \{0\}$. Since $S(X)$ is pointed irreducible, $\Phi' = \Phi$ by Proposition A.3.2 (see Appendix A). □

2.2. *Model category of Fréchet manifolds*

Let $U \subset X$, $V \in Y$ be open subsets of the Fréchet spaces X and Y, respectively, and $\psi : U \to V$ a map.

Definition 2.2.1. The derivative of $\psi : U \to V$ at a point $u \in U$ in the direction $x \in X$ is defined by

$$D^1\psi(u; x) = \lim_{\epsilon\to 0} \frac{\psi(u + \epsilon x) - \psi(u)}{\epsilon},$$

whenever the corresponding limit exists. One says that ψ is continuously differentiable or C^1 on U if the limit exists for all $u \in U$ and $x \in X$ and if the map

$$D^1\psi : U \times X \longrightarrow Y$$

is jointly continuous (as a map on a subset of the product).

Definition 2.2.2. The higher-order derivatives ($k \geq 2$) are inductively defined by

$$\begin{aligned}&D^k\psi(u; x_1, \ldots, x_k) \\ &\quad= \lim_{\epsilon\to 0} \frac{D^{k-1}\psi(u + \epsilon x_k; x_1, \ldots, x_{k-1}) - D^{k-1}\psi(u; x_1, \ldots, x_{k-1})}{\epsilon},\end{aligned}$$

whenever the corresponding limit exists.

One says ψ is C^k if $D^k\psi(u; x_1, \dots, x_k)$ exists for all $u \in U$ and $x_1, \dots, x_k \in X$ and is jointly continuous as a map

$$D^k\psi : U \times \underbrace{X \times \cdots \times X}_{k} \to Y.$$

A map ψ is smooth (C^∞) if it is C^k for all $k > 0$.

If ψ is C^k then $D^k\psi(u; x_1, \dots, x_k)$ is totally symmetric and linear separately in $x_1, \dots, x_k$ [16]. It can therefore be extended in the second variable to the map

$$D^k\psi : U \times S^k(X) \ni (u; x_1 \cdots x_k) \longrightarrow D^k\psi(u; x_1, \dots, x_k) \in Y,$$

where $S^k(X)$ is the kth symmetric tensor power of X. In the following the same symbol $D^k\psi$ will be used for the derivatives and for their extensions defined above.

Using the chain rule and the Leibniz rule for the first derivative [16] as well as an induction on k one gets the following:

Proposition 2.2.1. *Let U, V, W be open subsets of Fréchet space X, Y, Z, respectively, and $\phi : U \to V$, $\psi : V \to W$, C^k maps. Then the composition $\psi \circ \phi$ is a C^k map and for all $1 \le l \le k$, $u \in U$, and $x_1, \dots, x_k \in X$ one has*

$$D^l(\psi \circ \phi)(u; x_1, \dots, x_l) = \sum_{i=1}^{l} \frac{1}{i!} \sum_{\substack{\{P_1,\dots,P_i\} \\ |P_i|>0}} D^i\psi(\phi(u); D^{|P_1|}\phi(u; x_{P_1}), \dots, D^{|P_i|}\phi(u; x_{P_i})), \tag{4}$$

where the sum runs over all ordered nonempty i-partitions of the index set $\{1, \dots, l\}$.

Proposition 2.2.2. *For all C^k functions $f, g : U \to \mathbb{R}$, $1 \le l \le k$ and $x_1, \dots, x_l \in X$ one has*

$$D^l(f \cdot g)(u; x_1, \dots, x_k) = \sum_{\{P_1,P_2\}} D^{|P_1|} f(u; x_{P_1}) \cdot D^{|P_2|} g(u; x_{P_2}), \tag{5}$$

where the sum runs over all ordered two-partitions of the index set $\{1, \dots, l\}$ and the convention $D^0 f(u; x_\emptyset) \equiv f(u)$ is used.

Definition 2.2.3. The objects of the model category **fm** of smooth. Fréchet manifolds are open subsets $U \subset X$ where X runs over the category of Fréchet spaces.

For any two objects $U, V \in$ **Ofm** the space of morphisms **Mfm**(U, V) consists of all smooth maps $\psi : U \to V$. The composition of morphisms is defined as the composition of maps. An isomorphism in the category **fm** is called a diffeomorphism.

We denote by **fm**$^<$ the subcategory of **fm** consisting of all open subsets of finite-dimensional Fréchet spaces and all **fm**-morphism between them.

Let $\mathcal{C}_U^\infty = (U, \mathcal{C}^\infty(\cdot))$ be the sheaf of smooth functions on U. A smooth map $\psi : U \to V$ induces a morphism of sheaves of commutative algebras

$$\overline{\psi} = (\psi, \psi^*_\cdot) : \mathcal{C}_U^\infty \longrightarrow \mathcal{C}_V^\infty,$$

where for each open $V' \subset V$ the algebra map $\psi^*_{V'}$ is given by

$$\psi^*_{V'} : \mathcal{C}^\infty(V') \ni f \longrightarrow f \circ \psi \in \mathcal{C}^\infty(\psi^{-1}(V')).$$

A morphism $F = (F^0, F_\cdot) : \mathcal{C}_U^\infty \to \mathcal{C}_V^\infty$ of sheaves of commutative algebras is not in general of this form. However, for the finite-dimensional Fréchet spaces one has the following proposition.

Proposition 2.2.3. *Let $U \in \mathbb{R}^m$, $V \in \mathbb{R}^{m'}$ be open subsets.*
1. *For any morphism of sheaves of algebras*

$$F = (F^0, F_\cdot) : \mathcal{C}_U^\infty \longrightarrow \mathcal{C}_V^\infty,$$

$F^O : U \to V$ is smooth and $F = \overline{F^0}$.
2. *For any algebra morphism $A : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(U)$ there exists a unique smooth map $\psi : U \to V$ such that $\psi^*_U = A$.*

It follows that for $U, V \in \mathbf{Ofm}^<$ one has the 1–1 correspondence

$$M\mathbf{fm}(U, V) = M\mathbf{fm}^<(U, V) \ni \psi \longrightarrow \psi^*_V \in \mathrm{Alg}(\mathcal{C}^\infty(V), \mathcal{C}^\infty(U)).$$

This simple algebraic description of morphisms either as morphisms of sheaves of algebras or as morphisms of algebras of function is no longer valid for infinite-dimensional Fréchet spaces. In this case the space of morphisms of sheaves of algebras is essentially bigger than the space of smooth maps of open sets. In order to characterize the sheaf morphisms corresponding to smooth maps one needs some additional not algebraical conditions. This makes the idea of ringed spaces in infinite-dimensional geometry rather awkward and difficult to deal with [37,38]. This is also the main difficulty in developing a working infinite-dimensional extension of the Berezin–Leites–Kostant theory of supermanifolds which was originally developed as a theory of ringed spaces [8,19].

2.3. Model category of BLK supermanifolds

Definition 2.3.1. The objects of the model category **sm** of supermanifolds are sheaves of $\mathbb{Z}_2$-graded algebras

$$\mathcal{S}_U^{m,n} = (U, \mathcal{C}^\infty(\cdot) \otimes \wedge(\mathbb{R}^n)'),$$

where m, n are arbitrary nonnegative integers and U runs over all open sets of $\mathbb{R}^m$.

For any two objects $\mathcal{S}_U^{m,n}, \mathcal{S}_V^{m',n'} \in \mathbf{Osm}$ the space of morphisms $\mathbf{Msm}(\mathcal{S}_U^{m,n}, \mathcal{S}_V^{m',n'})$ consists of all morphisms of sheaves of $\mathbb{Z}_2$-graded algebras. The composition of morphisms is defined as the composition of morphisms of sheaves. An isomorphism in **sm** is called a superdiffeomorphism.

$(\mathbb{R}^n)'$ in the definition above denotes the space dual to $\mathbb{R}^n$. An object $\mathcal{S}_U^{m,n} \in$ **Osm** is called a *superdomain* of the superspace $\mathbb{R}^m \oplus \mathbb{R}^n$. For each open subset U' of the *underlying set* U the elements of the $\mathbb{Z}_2$-graded algebra $\mathcal{S}^{m,n}(U') = C^\infty(U') \otimes \wedge(\mathbb{R}^n)'$ are *superfunctions on* U'. According to the $\mathbb{Z}_2$-grading

$$\mathcal{S}^{m,n}(U) = \mathcal{S}^{m,n}(U)_0 \oplus \mathcal{S}^{m,n}(U)_1,$$

each superfunctions can be uniquely represented as the sum $f = f_0 + f_1$ of the *even* f_0 and the *odd* f_1 *parts.* A superfunction f is called *even (odd)* if $f_1 = 0\,(f_0 = 0)$.

For each **sm**-morphism $F = (F^0, F) : \mathcal{S}_U^{m,n} \to \mathcal{S}_V^{m',n'}$ the map $F^0 : U \to V$ is called the *underlying part of* F. In our notation, $F.$ denotes the family of $\mathbb{Z}_2$-graded algebra morphisms $F_{V'} : \mathcal{S}^{m',n'}(V') \to \mathcal{S}_V^{m,n}(F^{0-1}(V'),$ where V' runs over all open subsets of V. As a "super" counterpart of Proposition 2.2.3 one has [7,22,35].

Proposition 2.3.1. *Let* $\mathcal{S}_U^{m,n}, \mathcal{S}_V^{m',n'}$ *be superdomains.*
1. *For any* **sm**-*morphism* $F = (F^0, F) : \mathcal{S}_U^{m,n} \to \mathcal{S}_V^{m',n'}$ *the underlying map* $F^0 : U \to V$ *is smooth.*
2. *For any morphism* $A : \mathcal{S}^{m',n'}(V) \to \mathcal{S}^{m,n}(U)$ *of* $\mathbb{Z}_2$-*graded algebras there exists a unique* **sm**-*morphism* $F = (F^0, F) : \mathcal{S}_U^{m,n}, \mathcal{S}_V^{m',n'}$ *such that* $A = F_V$.

Remark 2.3.1. It follows from Propositions 2.2.3 and 2.3.1 that the covariant functor

$$\mathbf{Osm} \ni \mathcal{S}_U^{m,n} \longrightarrow U \in \mathbf{Ofm}^<,$$
$$\mathbf{Msm} \ni F = (F^0, F) \longrightarrow F^0 \in \mathbf{Mfm}^<$$

has the right inverse

$$\mathbf{Ofm}^< \ni U \longrightarrow C_U^\infty = \mathcal{S}_U^{m,0} \in \mathbf{Osm},$$
$$\mathbf{Mfm}^< \ni \psi \longrightarrow \bar{\psi} = (\psi, \psi^*) \in \mathbf{Msm}. \tag{6}$$

The image of the functor above coincides with the subcategory $\mathbf{sm}_0$ of **sm** consisting of all objects of the form $\mathcal{S}_U^{m,0}$ and all **sm**-morphisms between them. It follows that the model category of finite-dimensional manifolds $\mathbf{fm}^<$ can be regarded as the full subcategory $\mathbf{sm}_0$ of the model category of BLK supermanifolds.

Let $\{\bar{u}_\mu\}_{\mu=1}^m$ be a standard basis in $\mathbb{R}^m$. The functions $u^\mu : \mathbb{R}^m \supset U \ni u \to u^\mu \in \mathbb{R}$ uniquely defined by $u = \Sigma_{\mu=1}^m u^\mu \bar{u}_\mu$ are called the *standard coordinates* on $U \subset \mathbb{R}^m$. Let $\{\theta^\alpha\}_{\alpha=1}^n$ be the standard basis in $(\mathbb{R}^n)'$. The collection $\{u^1, \ldots, \theta^1, \ldots, \theta^n\}$ regarded as a subset of $\mathcal{S}^{m,n}(U) = C^\infty(U) \otimes \wedge(\mathbb{R}^n)'$ is called the *standard coordinate system on* $\mathcal{S}_U^{m,n}$. In the standard coordinates every superfunction $f \in \mathcal{S}^{m,n}(U)$ has a unique representation

$$f = f(u, \theta) = f^0(u) + f_0^\wedge(u, \theta) + f_1^\wedge(u, \theta), \tag{7}$$

where

$$f_0^\wedge(u,\theta)=\sum_{\substack{k=2\\ \text{even}}}^{n}\frac{1}{k!}\sum_{\alpha_1,\ldots,\alpha_k=1}^{n} f^\wedge_{\alpha_1\ldots\alpha_k}(u)\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_k},$$

$$f_1^\wedge(u,\theta)=\sum_{\substack{k=1\\ \text{odd}}}^{n}\frac{1}{k!}\sum_{\alpha_1,\ldots,\alpha_k=1}^{n} f^\wedge_{\alpha_1\ldots\alpha_k}(u)\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_k},$$

and $f^0(u)$, $f^\wedge_{\alpha_1\ldots\alpha_k}(u)\in\mathcal{C}^\infty(U)$. The coefficient functions $f^\wedge_{\alpha_1\ldots\alpha_k}(u)$ are assumed to be totally antisymmetric in their indices. In the decomposition (7), $f^0(u)$ is called the *underlying* and $f^\wedge(u,\theta) = f_0^\wedge(u,\theta)+f_1^\wedge(u,\theta)$ the *exterior part* of the superfunction f.

Note that for an arbitrary **sm**-morphism $F=(F^0,F):\mathcal{S}_U^{m,n}\to\mathcal{S}_V^{m',n'}$ and any superfunction $g\in\mathcal{S}^{m',n'}(V)$ one has

$$(F_V g)^0 = F_V(g^0) = g^0\circ F^0,\qquad (F_V g)_0^\wedge = F_V(g_0^\wedge),$$

but in general $(F_V g)_1^\wedge \neq F_V(g_1^\wedge)$.

Definition 2.3.2. Let $\{u^1,\ldots,u^m,\theta^1,\ldots,\theta^n\}$ be the standard coordinate system on $\mathcal{S}_U^{m,n}$. For each open subset $U'\subset U$ the partial derivatives of a superfunction $f\in\mathcal{S}^{m,n}(U')$ are defined by

$$\frac{\partial}{\partial u^\mu}f(u,\theta)=\frac{\partial}{\partial u^\mu}f^0(u)+\sum_{k=1}^{n}\frac{1}{k!}\sum_{\alpha_1,\ldots,\alpha_k=1}^{n}\frac{\partial}{\partial u^\mu}f^\wedge_{\alpha_1\ldots\alpha_k}(u)\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_k},$$

$$\frac{\partial}{\partial\theta^\alpha}f(u,\theta)=\sum_{k=1}^{n}\frac{1}{k!}\sum_{\alpha_1,\ldots,\alpha_k=1}^{n}\sum_{i=1}^{k}(-1)^{i+1}\delta_{\alpha\alpha_i}f_{\alpha_1\alpha_2\ldots\alpha_k}(u)$$
$$\times\,\theta^{\alpha_1}\wedge\cdots\wedge\widehat{\theta^{\alpha_i}}\wedge\cdots\wedge\theta^{\alpha_k},$$

where $\widehat{\theta^{\alpha_i}}$ means that θ^{α_i} is omitted.

In the following we shall also use the compact notation $\{x^I\}_{I=1}^{m+n}$ for the standard coordinate system $\{u^\mu\}_{\mu=1}^m\cup\{\theta^\alpha\}_{\alpha=1}^n$ on $\mathcal{S}_U^{m,n}$, where

$$x^I=\begin{cases}u^I & \text{for } I=1,\ldots,m\\ \theta^{I-m} & \text{for } I=m+1,\ldots,m+n.\end{cases}$$

Accordingly, for the partial derivatives one has

$$\partial_I=\begin{cases}\dfrac{\partial}{\partial u^I} & \text{for } I=1,\ldots,m,\\[2ex] \dfrac{\partial}{\partial\theta^{I-m}} & \text{for } I=m+1,\ldots,m+n.\end{cases}$$

Let $\{\bar\theta_\alpha\}_{\alpha=1}^n$ be the basis in $\mathbb{R}^n$ dual to the basis $\{\theta^\alpha\}_{\alpha=1}^n$. Then $\{\bar x_I\}_{I=1}^{m+n}=\{\bar u_\mu\}_{\mu=1}^m\cup\{\bar\theta_\alpha\}_{\alpha=1}^n$ is the standard basis in the $\mathbb{Z}_2$-graded space $\mathbb{R}^m\oplus\mathbb{R}^n$.

Definition 2.3.3. Let $f \in \mathcal{S}^{m,n}(U)$ and $a = \Sigma a^I \bar{x}_I \in \mathbb{R}^m \oplus \mathbb{R}^n$. The first derivative of the superfunction f in the direction a is defined by

$$D^1 f(u,\theta;a) = \sum_{I=1}^{m+n} a^I \partial_I f(u,\theta).$$

For $a_1,\ldots,a_k \in \mathbb{R}^m \oplus \mathbb{R}^n$ the higher order directional derivatives are given by

$$D^k f(u,\theta;a_1,\ldots,a_k) = \sum_{I_1,\ldots,I_k=1}^{m+n} a_1^{I_1}\cdots a_k^{I_k}\partial_{I_1}\cdots\partial_{I_k} f(u,\theta).$$

The kth-order differentiations ($k \geq 1$) are defined as the underlying parts of the kth-order directional derivatives:

$$\tilde{D}^k f(u;a_1,\ldots,a_k) = \sum_{I_1,\ldots,I_k=1}^{m+n} a_1^{I_1}\cdots a_k^{I_k}(\partial_{I_1}\cdots\partial_{I_k} f)^0(u).$$

The higher directional derivatives and differentiations are totally antisymmetric in the variables $a_1,\ldots,a_k$ and therefore can be uniquely extended (in the second variable) to linear functions on the kth graded symmetric tensor power $S^k(\mathbb{R}^m \oplus \mathbb{R}^n)$ of the $\mathbb{Z}_2$-graded space $\mathbb{R}^m \oplus \mathbb{R}^n$. With this interpretation one can use the notation of Remark 2.1.2 for arguments of multilinear functions.

Using the graded Leibnitz rule for first derivatives and induction on k one gets the following multiple Leibnitz rule for superfunctions.

Proposition 2.3.2. *Let $f, g \in \mathcal{S}^{m,n}(U)$ be superfunctions and $\mathcal{X} = \{a_i\}_{i=1}^k$ a sequence of homogeneous elements of the graded space $\mathbb{R}^m \oplus \mathbb{R}^n$. Then*

$$D^k(f\cdot g)(u,\theta;a_1,\ldots,a_k) = \sum_{\mathcal{P}=\{P_1,P_2\}} \sigma(\mathcal{X},\mathcal{P})(-1)^{|f||a_{P_2}|} D^{|P_1|} f(u,\theta;a_{P_1})\cdot D^{|P_2|} g(u,\theta;a_{P_2}), \tag{8}$$

where the sum runs over all 2-partitions of the index set $\{1,\ldots,k\}$, and the notation of Remark 2.1.2 *as well as the convention $D^0 f(u,\theta;x_\emptyset) \equiv f(u,\theta)$ are used.*

The standard coordinate system on $\mathcal{S}_U^{m,n}$ generates the subalgebra of superfunctions with polynomial coefficients. With the topology of uniform convergence on compact subsets this is a dense subalgebra of $\mathcal{S}^{m,n}(U)$. The following proposition [7,22,25,35] says that the standard coordinates behave as algebraic generators with respect to $\mathbb{Z}_2$-graded algebra morphisms. This property is essential for the coordinate description of morphisms in **sm**.

Proposition 2.3.3. *Let $\{x^I\}_{I=1}^{m+n}$ and $\{y^J\}_{J=1}^{m'+n'}$ be the standard coordinate systems on $\mathcal{S}_U^{m,n}$ and $\mathcal{S}_V^{m',n'}$, respectively. Let $\{F^J\}_{J=1}^{m'+n'}$ be a collection of supperfunctions on $\mathcal{S}_U^{m,n}$ such* *that F^J is even for $J = 1, \dots, m'$, F^J is odd for $J = m'+1, \dots, m'+n'$, and the map*

$$F^0 : U \ni u \to (F^{10}(u), \dots, F^{m'0}(u)) \in \mathbb{R}^{m'}$$

has its image in $V \subset \mathbb{R}^{m'}$.

Then there exists a unique **sm**-*morphism $F = (F^0, F) : \mathcal{S}_U^{m,n} \to \mathcal{S}_V^{m',n'}$ such that*

$$F_V(y^J) = F^J(u, \theta), \quad J = 1, \dots, m'+n'. \tag{9}$$

The superfunctions $F^J(u,\theta) = F_V(y^J)$ are called the *coordinate representation* of the **sm**-morphism $F = (F^0, F)$. Formula (9) is frequently written in the following somewhat incorrect but more intuitive form:

$$v^\nu = F^{\nu 0}(u) + \sum_{\substack{k=2\\ \text{even}}}^{n} \frac{1}{k!} \sum_{\alpha_1,\dots,\alpha_k=1}^{n} F^{\nu\wedge}_{\alpha_1\cdots\alpha_k}(u)\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_k}, \quad \nu = 1, \dots, m',$$

$$v^\beta = \sum_{\substack{k=1\\ \text{odd}}}^{n} \frac{1}{k!} \sum_{\alpha_1,\dots,\alpha_k=1}^{n} F^{\beta\wedge}_{\alpha_1\cdots\alpha_k}(u)\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_k}, \quad \beta = 1, \dots, n'.$$

Proposition 2.3.4. *Let $\{x^I\}_{I=1}^{m+n}$ and $\{y^J\}_{J=1}^{m'+n'}$ be standard coordinate systems on $\mathcal{S}_U^{m,n}$ and $\mathcal{S}_V^{m',n'}$, respectively. Let $F = (F^0, F) : \mathcal{S}_U^{m,n} \to \mathcal{S}_V^{m',n'}$ be an* **sm**-*morphism with the coordinate representation $\{F^J(u,\theta)\}_{J=1}^{m'+n'}$.*

Then for any superfunction $g \in \mathcal{S}^{m',n'}(V)$, and any sequence $\mathcal{X} = \{a_1, \dots, a_k\}$ of homogeneous elements of $\mathbb{R}^m \oplus \mathbb{R}^n$ one has $(F_V g)^0(u) = g^0 \circ F^0(u)$, and

$$\tilde{D}^k(F_V g)(u; a_1, \dots, a_k) = \sum_{l=1}^{k} \frac{1}{l!} \sum_{\substack{\{P_1,\dots,P_l\}\\ P_i \neq \emptyset}} \sigma(\mathcal{X}, \mathcal{P}) \tilde{D}^l g(F^0(u), \tilde{D}^{|P_1|}F(u, a_{P_1}), \dots, \tilde{D}^{|P_l|}F(u, a_{P_l})), \tag{10}$$

where the sum runs over all nonempty partitions of the index set $\{1, \dots, k\}$, and for each $u \in U$,

$$\tilde{D}^{|P_1|}F(u, a_{P_1}) = \sum_{J=1}^{m'+n'} \tilde{D}^{|P_1|}F^J(u, a_{P_1})\tilde{y}_J \in \mathbb{R}^{m'} \oplus \mathbb{R}^{n'}.$$

Proof. For all even $a_1, \dots, a_k \in \mathbb{R}^n \oplus \{o\}$ formula (10) is the multiple chain rule for Fréchet manifolds (Proposition 2.2.1) applied to the function $g^0 \circ F^0(u)$.

For all odd $a_1, \dots, a_k \in \{o\} \oplus \mathbb{R}^m$ formula (10) is equivalent to the standard Taylor expansion for the pull-back of a superfunction [7,22,25,35]. Indeed, let

$$F_V g(u,\theta) = F_V g^0(u) + \sum_{k=1}^{n} \frac{1}{k!} \sum_{\alpha_1,\dots,\alpha_k=1}^{n} F_V g^{\wedge}_{\alpha_1\cdots\alpha_k}(u)\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_k}.$$

be the coordinate expression for the superfunction $F_V g$, and $\{\bar{\theta}_\alpha\}_{\alpha=1}^n$ the standard basis in $\mathbb{R}^n$. Then the coefficients of the representation above are given by

$$F_V g^{\wedge}_{\alpha_k \cdots \alpha_1}(u) = \tilde{D}^k F_V g(u; \bar{\theta}_{\alpha_1}, \dots, \bar{\theta}_{\alpha_k})$$
$$= \sum_{l=1}^{k} \frac{1}{l!} \sum_{\substack{\{P_1,\dots,P_l\} \\ P_i \neq \emptyset}} \sigma(\mathcal{X}, \mathcal{P}) \tilde{D}^l g(F^0(u), F^{\wedge}_{P_1}(u), \dots, f^{\wedge}_{P_l}(u)), \tag{11}$$

where $\mathcal{X} = \{\bar{\theta}_{\alpha_1}, \dots, \bar{\theta}_{\alpha_k}\}$ and for each subset $P = \{\alpha_1, \dots, \alpha_i\}(\alpha_1 < \cdots < \alpha_i)$ of the index set $\{1, \dots, k\}$

$$F^{\wedge}_P(u) = \sum_{J=1}^{m'+n'} \tilde{D}^{|P|} F^{J\wedge}(u, \bar{\theta}_{\alpha_1}, \dots, \bar{\theta}_{\alpha_i}) \bar{y}_J$$
$$= \sum_{J=1}^{m'+n'} F^{J\wedge}_{\alpha_i,\dots,\alpha_1}(u) \bar{y}_J.$$

The general case can be derived from formula (11) by differentiating in even directions. □

Formulae (10) and (11) suggest the following slightly modified description of an **sm**-morphism.

Definition 2.3.4. Let $\{F^J(u, \theta)\}_{J=1}^{m'+n'}$ be the coordinate representation of an **sm**-morphism $F = (F^0, F_.) : \mathcal{S}^{m,n}_U \to \mathcal{S}^{m',n'}_V$. For every $k \geq 1$ the kth infinitesimal component of $F = (F^0, F_.)$ is defined by

$$F^+_k : U \times S^k(\mathbb{R}^m \oplus \mathbb{R}^n) \ni (u, a_1 \dots a_k)$$
$$\longrightarrow \sum_{J=1}^{m'+n'} \tilde{D}^k F^J(u; a_1, \dots, a_k) \bar{y}_J \in \mathbb{R}^{m'} \oplus \mathbb{R}^{n'}.$$

The restriction of the map F^+_k to $U \times \wedge^k \mathbb{R}^n \subset U \times S^k(\mathbb{R}^m \oplus \mathbb{R}^n)$:

$$F^{\wedge}_k : U \times \wedge^k \mathbb{R}^n \ni (u, \xi_1 \dots \xi_k) \longrightarrow \sum_{J=1}^{m'+n'} \tilde{D}^k F^J(u; \xi_1, \dots, \xi_k) \bar{y}_J \in \mathbb{R}^{m'} \oplus \mathbb{R}^{n'}$$

is called the kth exterior component of the **sm**-morphism $F = (F^0, F_.)$.

As a simple consequence of the definition and Proposition 2.3.3 one has the following:

Proposition 2.3.5. *Let $\{x^I\}_{I=1}^{m+n}$ and $\{y^J\}_{J=1}^{m'+n'}$ be standard coordinate systems on $\mathcal{S}^{m,n}_U$ and $\mathcal{S}^{m',n'}_V$, respectively. Let $\Phi^0 : \mathbb{R}^m \supset U \to V \subset \mathbb{R}^{m'}$ be a smooth map and $\{\Phi^{\wedge}_k\}_{k=1}^n$ a family of smooth maps*

$$\Phi_k^\wedge : U \times \wedge^k \mathbb{R}^n \longrightarrow \mathbb{R}^{m'} \oplus \mathbb{R}^{n'},$$

linear and even in the second variable.

Then there exists a unique **sm**-*morphism* $F = (F^0, F) : \mathcal{S}_U^{m,n} \to \mathcal{S}_V^{m',n'}$ *with underlying part* $F^0 = \Phi^0$ *and exterior components* $F_k^\wedge = \Phi_k^\wedge; k = 1, \dots, n$.

Remark 2.3.2. The underlying part and the exterior components of an **sm**-morphism F contain essentially the same data as the coordinate representation of F. The virtue of the description in terms of exterior components is that it is independent of the choice of basis in the model superspace and as we shall see in the Section 2.4 that it can be easily generalized to the infinite-dimensional case. Here we shall consider composition of **sm**-morphisms is this language.

Let $G = (G^0, G.) : \mathcal{S}_V^{m',n'} \to \mathcal{S}_W^{m'',n''}$ be another **sm**-morphism, and $\{z^K\}_{K=1}^{m''+n''}$ the standard coordinate system on $\mathcal{S}_W^{m'',n''}$. For the underlying parts one has

$$(G \circ F)^0 = G^0 \circ F^0.$$

The coordinate representation of the composition $G \circ F : \mathcal{S}_U^{m,n} \to \mathcal{S}_W^{m'',n''}$ is given by

$$(G \circ F)^K = (G \circ F)_W(z^K) = F_V(G_W(z^K)) = G_V(F^K).$$

Calculating the RHS by formula (11) and using Definition 2.3.4 one gets the following expression for the exterior components of the composition

$$(G \circ F)^\wedge(u, \xi_1, \dots, \xi_k) = \sum_{l=1}^{k} \frac{1}{l!} \sum_{\substack{\{P_1,\dots,P_l\} \\ P_i \neq \emptyset}} \sigma(\mathcal{X}, \mathcal{P}) G^+(F^0(u), F^\wedge(u, \xi_{P_1}), \dots, F^\wedge(u, \xi_{P_l})). \tag{12}$$

Let us note that the exterior components of the composition depend on the infinitesimal components G_k^+ of G and therefore involve partial derivatives in even directions of the exterior components $F_k^\wedge$. This property of composition of **sm**-morphisms is responsible for most of the peculiar features of supergeometry. In particular, this is the reason for which smooth structures and Fréchet spaces are indispensable.

2.4. BLK supermanifolds

Definition 2.4.1. A supermanifold modelled on the superspace $\mathbb{R}^m \oplus \mathbb{R}^n$ is a sheaf $\mathcal{A}_M = (M, \mathcal{A}(\cdot))$ of $\mathbb{Z}_2$-graded algebras on a Hausdorff topological space M such that for each $p \in M$ there exist an open neighbourhood U of p, and an isomorphism of sheaves of $\mathbb{Z}_2$-graded algebras

$$F = (F^0, F.) : \mathcal{A}_U \longrightarrow \mathcal{S}_{\Phi^0(U)}^{m,n},$$

where $\mathcal{A}_U = (U, \mathcal{A}(\cdot))$ is the restriction of $\mathcal{A}_M$ to U, and $\mathcal{S}_{\Phi^0(U)}^{m,n}$ is a superdomain, i.e. $\Phi^0(U)$ is an open subset of $\mathbb{R}^m$ and $\mathcal{S}_{\Phi^0(U)}^{m,n} = (\Phi^0(U), \mathcal{C}^\infty(\cdot) \otimes \wedge(\mathbb{R}^n)')$.

Definition 2.4.2. The objects of the category **SM** are supermanifolds modelled on superspaces $\mathbb{R}^m \oplus \mathbb{R}^n$ where $m, n \geq 0$.

For any two objects $\mathcal{A}_M, \mathcal{B}_N \in$ **OSM** the space of morphisms **MSM** consists of all morphisms of sheaves of $\mathbb{Z}_2$-graded algebras. The composition of **SM**-morphisms is defined as the composition of morphisms of sheaves.

Let $\mathcal{A}_M$ be a supermanifold. The collection $\{(U_\alpha, F_\alpha)\}_{\alpha\in I}$ of isomorphisms of sheaves of $\mathbb{Z}_2$-graded algebras

$$F_\alpha = (F^0_\alpha, F_\alpha) : \mathcal{A}_{U_\alpha} \longrightarrow \mathcal{S}^{m,n}_{F^0_{\alpha.}(U_\alpha)}$$

such that $\{U_\alpha\}_{\alpha\in I}$ is an open cover of M is called an $(m, n) - atlas$ on $\mathcal{A}_M$.

Let $\{(U_\alpha, F_\alpha)\}_{\alpha\in I}$ be an atlas on a supermanifold $\mathcal{A}_M$. Then by Remark 2.3.1 the collection $\{F^0_\alpha\}_{\alpha\in I}$ is a smooth atlas on M. By the same token different atlases on $\mathcal{A}_M$ lead to compatible atlases and M acquires a unique smooth structure. M with this structure is called the *underlying manifold* of $\mathcal{A}_M$.

One has the following global version of Proposition 2.3.1 [7,22,25,35].

Proposition 2.4.1. *Let $\mathcal{A}_M, \mathcal{B}_N$ be supermanifolds.*
1. *For any* **SM**-*morphism $F = (F^0, F_.) : \mathcal{A}_M \to \mathcal{B}_N$ the underlying map $F^0 : M \to N$ is smooth.*
2. *For any morphism $A : \mathcal{B}(N) \to \mathcal{A}(M)$ of $\mathbb{Z}_2$-graded algebras there exists a unique* **SM**-*morphism $F = (F^0, F_.) : \mathcal{A}_M \to \mathcal{B}_N$ such that $A = F_N$.*

Let $F = (F^0, F_.) : \mathcal{A}_M \to \mathcal{B}_N$ be a morphism of supermanifolds and $\{(V_\gamma, G_\gamma)\}_{\gamma\in J}$ an atlas on $\mathcal{B}_N$. Then there exsists an atlas $\{(U_\alpha, F_\alpha)\}_{\alpha\in I}$ on $\mathcal{A}_M$ such that for all $\alpha \in I$ there exists (a nonnecessarily unique) $\alpha' \in J$ for which $F^0(U_\alpha) \subset V_{\alpha'}$. The family of **sm**-morphisms $\{F_{\alpha\alpha'}\}_{\alpha\in I}$ defined for each $\alpha \in I$ by

$$F_{\alpha\alpha'} = G_{\alpha'} \circ F \circ F^{-1}_\alpha$$

is called *a representation of the morphism* $F = (F^0, F_.) : \mathcal{A}_M \to \mathcal{B}_N$ in the atlas $\{(V_\gamma, \Psi_\gamma)\}_{\gamma\in J}$ on $\mathcal{B}_N$.

Proposition 2.4.2. *Let $\{(U_\alpha, F_\alpha)\}_{\alpha\in J}$ be an (m, n)-atlas on a supermanifold $\mathcal{A}_M$, and $\{(V_\gamma, G_\gamma)\}_{\gamma\in J}$ an (m', n')-atlas on a supermanifold $\mathcal{B}_N$. Let $\{F_{\alpha\alpha'}\}_{\alpha\in I}$ be a family of maps such that:*
1. *for all $\alpha \in I$, $F_{\alpha\alpha'} : \mathcal{S}^{m,n}_{F^0_\alpha(U_\alpha)} \to \mathcal{S}^{(m',n')}_{G^0_{\alpha'}(V_{\alpha'})}$ is an* **sm**-*morphism;*
2. *for all $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ one has*

$$F_{\alpha\beta'} = G_{\beta'} \circ G^{-1}_{\alpha'} \circ F_{\alpha\alpha'} \circ F_\alpha \circ F^{-1}_\beta .$$

Then there exists a unique smooth morphism $F = (F^0, F_.) : \mathcal{A}_M \to \mathcal{B}_N$ of supermanifolds such that $\{F_{\alpha\alpha'}\}_{\alpha\in I}$ is a representation of F in the atlas $\{(V_\gamma, G_\gamma)\}_{\gamma\in J}$.

Definition 2.4.3. Let $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ be an admissible atlas on a smooth manifold M of dimension m. Let $\{F_{\alpha\beta}\}_{\alpha\in I}$ be a collection of maps such that
1. for all $\alpha \in I$, $\beta \in I(\alpha) \equiv \{\beta \in I : U_\alpha \cap U_\beta \neq \emptyset\}$

$$F_{\alpha\beta} = (F^0, F_{\alpha\beta}.) : \mathcal{S}^{m,n}_{\varphi_\alpha(U_\alpha\cap U_\beta)} \longrightarrow \mathcal{S}^{m,n}_{\varphi_\beta(U_\alpha\cap U_\beta)}$$

is an isomorphism of superdomains such that $F^0_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$;
2. for all $\alpha \in I$, $F_{\alpha\alpha} = \mathrm{id}_{\mathcal{S}^{m,n}_{\varphi_\alpha(U_\alpha)}}$;
3. for all $\alpha, \beta, \gamma \in I$ such that $U_\alpha \cap U_\beta \cap U\gamma \neq \emptyset$, $F_{\beta\gamma} \circ F_{\alpha\beta} = F_{\alpha\gamma}$ on $\mathcal{S}^{m,n}_{\varphi_\alpha(U_\alpha\cap U_\beta\cap U_\gamma)}$.

A collection $\{F_{\alpha\beta}\}_{\alpha\in I}$ with the properties stated above is called an (m, n)-cocycle of transition **sm**-morphisms over the atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ on M.

Two cocycles $\{F'_{\alpha'\beta'}\}_{\alpha'\in I}$, $\{F''_{\alpha''\beta''}\}_{\alpha''\in I''}$ of transition **sm**-morphisms on M are said to be *compatible* if there exists a third one $\{F_{\alpha\beta}\}_{\alpha\in I}$ such that

$$\{\varphi'_{\alpha'}\}_{\alpha'\in I'} \cup \{\varphi''_{\alpha''}\}_{\alpha''\in I''} \subset \{\varphi_\alpha\}_{\alpha\in I},$$
$$\{F'_{\alpha'\beta'}\}_{\alpha'\in I'} \cup \{F''_{\alpha''\beta''}\}_{\alpha''\in I''} \subset \{F_{\alpha\beta}\}_{\alpha\in I},$$

as sets of maps.

Proposition 2.4.3. *Let $\{F_{\alpha\beta}\}_{\alpha\in I}$ be an (m, n)-cocyle of transition* **sm***-morphisms on M. Then there exists a unique supermanifold $\mathcal{A}_M$ with the underlying manifold M and with the (m, n)-atlas $\{(U_\alpha, F_\alpha)\}_{\alpha\in I}$ such that*

$$F_{\alpha\beta} = F_\beta \circ F_\alpha^{-1}|_{F^0_\alpha(U_\alpha\cap U_\beta)}^{\mathcal{S}^{n,m}},$$

for all $\alpha \in I$, $\beta \in I(\alpha)$.

Compatible (m, n)-cocycles of transition **sm***-morphisms on M lead by the construction above to the same supermanifold $\mathcal{A}_M$.*

The supermanifold $\mathcal{A}_M$ and the (m, n)-atlas $\{(U\alpha, F_\alpha)\}_{\alpha\in I}$ of the proposition above are said to be *generated* by the (m, n)-cocycle $\{F_{\alpha\beta}\}_{\alpha\in I}$.

3. Model category

3.1. Objects

Let X be a vector space. The *group-like Hopf algebra $G(X)$ of X* is the free vector space $\mathbb{R}X$ of X (i.e. the vector space over $\mathbb{R}$ containing X as a basis) endowed with the trivial $\mathbb{Z}_2 \oplus \mathbb{Z}_+$ grading

$$\mathbb{R}X = \bigoplus_{\substack{k\geq 0\\ i=0,1}} \mathbb{R}X^k_i, \qquad \mathbb{R}X^k_i = \begin{cases} \mathbb{R}X & \text{for } k = i = 0,\\ \{o\} & \text{otherwise}, \end{cases}$$

and with the Hopf algebra structure given by

$$M_G(x \otimes y) = x + y, \qquad u_G(1) = \mathrm{o},$$
$$\Delta_G(x) = x \otimes x, \quad \varepsilon_G(x) = 1, \quad s_G(x) = -x$$

for all $x, y \in X$ (+, o, − stand for the addition, the zero vector, and the inverse in the vector space X). One can easily verify that $G(X)$ is a pointed bigraded commutative cocommutative Hopf algebra. The direct sum decomposition into irreducible components takes the form

$$G(X) = \bigoplus_{x \in X} \mathbb{R}x,$$

where $\mathbb{R}x$ is a one-dimensional subcoalgebra generated by the group-like element $x \in X \subset G(X)$. For every subset $U \subset X$ the free vector space $\mathbb{R}U$ is a bigraded subcoalgebra of $\mathbb{R}X$,

$$\mathbb{R}U = \bigoplus_{x \in U} \mathbb{R}x \subset \mathbb{R}X.$$

$\mathbb{R}U$ with the induced bigraded coalgebra structure is called the *group-like coalgebra of* U.

Definition 3.1.1. Let $X = X_0 \oplus X_1$ be a graded space. The tensor product

$$\mathcal{D}_X = G(X_0) \otimes S(X)$$

of the group-like Hopf algebra $G(X_0)$ of X_0 and the symmetric algebra $S(X)$ of X is called the Hopf algebra of the graded space X.

The Hopf algebra structure on $\mathcal{D}_X$ is given by

$$M_{\mathrm{H}}((u \otimes \alpha) \otimes (w \otimes \beta)) = (u + w) \otimes \alpha \cdot \beta,$$
$$u_{\mathrm{H}}(1) = \mathrm{o} \otimes 1,$$
$$\Delta_{\mathrm{H}}(u \otimes \alpha) = \sum_{(\alpha)} (u \otimes \alpha_{(1)}) \otimes (u \otimes \alpha_{(2)}),$$
$$\varepsilon_{\mathrm{H}}(u \otimes \alpha) = \varepsilon_G \otimes \varepsilon(u \otimes \alpha) = \varepsilon(\alpha),$$
$$s_{\mathrm{H}}(u \otimes \alpha) = s_G \otimes s(u \otimes \alpha) = (-u) \otimes s(\alpha),$$

for all $u, w \in X_0$; $\alpha, \beta \in S(X)$, where ε is given by Proposition 2.1.2, and s by Remark 2.1.4. By definition of $G(X_0)$ and Theorem 2.1.1, $\mathcal{D}_X$ is a pointed bigraded commutative cocommutative Hopf algebra. Since $G(X_0)$ is generated by group-like elements and $S(X)$ is pointed irreducible one has the following decomposition into irreducible components:

$$\mathcal{D}_X = \bigoplus_{u \in X_0} \mathcal{D}_{X_u},$$

where $\mathcal{D}_{X_u} = \mathbb{R}u \otimes S(X) \cong S(X)$ for all $u \in X_0$.

The subcoalgebra $\mathcal{D}_{X_{\mathrm{o}}}$ is a strictly bigraded Hopf subalgebra of $\mathcal{D}_X$. It acts on $\mathcal{D}_X$ by the left and right multiplication. In particular using the identification $\mathcal{D}_{X_{\mathrm{o}}} \cong S(X)$ one gets the right action of $S(X)$ on $\mathcal{D}_X$,

$$R : \mathcal{D}_X \otimes S(X) \ni (u \otimes \alpha) \otimes \beta \longrightarrow (u \otimes \alpha) \cdot \beta = u \otimes \alpha \cdot \beta \in \mathcal{D}_X.$$

By construction each irreducible component is stable with respect to R and for every $u \in X_0$ the map

$$R_u : S(X) \ni \alpha \longrightarrow (u \otimes 1) \cdot \alpha \in \mathcal{D}_{X_u}$$

is an isomorphism of bigraded coalgebras. For the sake of simplicity we will write u for the group like element $u \otimes 1 \in \mathcal{D}_X$. With this convention $X_0 \subset \mathcal{D}_X$ and $u \otimes \alpha = u \cdot \alpha$ for all $u \otimes \alpha \in \mathcal{D}_X$.

Definition 3.1.2. Let $\mathcal{D}_X$ be the Hopf algebra of a graded space $X = X_0 \oplus X_1$, and U a subset of X_0. The subcoalgebra $\mathcal{D}_X(U) \subset \mathcal{D}_X$

$$\mathcal{D}_X(U) = \bigoplus_{u \in U} \mathcal{D}_{X_u},$$

is called the subcoalgebra over U.

In the case of a graded Fréchet space, i.e. a topological direct sum $X = X_0 \oplus X_1$ of Fréchet spaces, the subcoalgebra $\mathcal{D}_X(U)$ over an open subset $U \subset X_0$ will be called an open subcoalgebra of $\mathcal{D}_X$.

By definition, for any $U \subset X_0$ the subcoalgebra $\mathcal{D}_X(U)$ over U is the tensor product of bigraded coalgebras $\mathcal{D}_X(U) = G(U) \otimes S(X)$. As the direct sum of irreducible components of $\mathcal{D}_X$ it is stable under the $S(X)$ right action.

3.2. Morphisms

Let X, Y be graded spaces and $\mathcal{D}_X(U), \mathcal{D}_Y(V)$ subcoalgebras over $U \subset X_0$ and $V \subset Y_0$, respectively. Let $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ be a morphism of graded coalgebras. Φ sends group-like elements into group-like elements and irreducible components into irreducible ones. It follows that Φ is uniquely determined by the map

$$U \times S(X) \ni u \otimes \alpha \to \Phi(u \otimes \alpha) \in V \times S(Y), \tag{13}$$

where the cartesian products $U \times S(X), V \times S(Y)$ are identified with the subsets of $\mathcal{D}_X(U), \mathcal{D}_Y(V)$ by the inclusions

$$U \times S(X) \ni (u, \alpha) \longrightarrow u \otimes \alpha = u \cdot \alpha \in \mathcal{D}_X(U),$$
$$U \times S(Y) \ni (u, \beta) \longrightarrow u \otimes \beta = v \cdot \beta \in \mathcal{D}_Y(V).$$

Note that the map (13) can be regarded as the family $\{\Phi_u\}_{u \in U}$ of $\mathbb{Z}_2$-graded coalgebra morphisms

$$\Phi_u : \mathcal{D}_{X_u} \ni u \cdot \alpha \longrightarrow \Phi(u \cdot \alpha) \in \mathcal{D}_{X\Phi(u)}.$$

Since each irreducible component of $\mathcal{D}_Y(V)$ is isomorphic to the bigraded coalgebra $S(Y)$ the universal property of the symmetric Hopf algebra can be used for a more detailed description of Φ. In order to study various properties of graded coalgebra morphisms it is covenient to introduce the following definition.

Definition 3.2.1. Let X, Y be $\mathbb{Z}_2$-graded spaces and $\mathcal{D}_X(U), \mathcal{D}_Y(V)$ subcoalgebras over $U \subset X_0$ and $V \subset Y_0$, respectively. Denote by $\pi : V \times S(Y) \to S(Y)$, resp. $\pi_Y : S(Y) \to Y$ the canonical projections on the second factor, resp. on $S^1(Y) = Y$. Let $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ be a morphism of $\mathbb{Z}_2$-graded coalgebras. The maps

$$\Phi^0 : U \ni u \longrightarrow \Phi(u) \in V,$$

$$\Phi^+ : U \times S(X)^+ \ni (u, \beta) \longrightarrow \pi_Y \circ \pi(\Phi(u \cdot \beta)) \in Y,$$

$$\Phi^\wedge : U \times \wedge(X_1)^+ \ni (u, \gamma) \longrightarrow \pi_Y \circ \pi(\Phi(u \cdot \gamma)) \in Y$$

and called the underlying, the infinitesimal, and the exterior parts of Φ, respectively. For every $k \geq 1$, the restrictions

$$\Phi_k^+ : U \times \underbrace{X \times \cdots \times X}_{k} \ni (u, a_1, \ldots, a_k) \longrightarrow \pi_Y \circ \pi(\Phi(u \cdot a_1 \cdots a_k)) \in Y,$$

$$\Phi_k^\wedge : U \times \underbrace{X_1 \times \cdots \times X_1}_{k} \ni (u, \xi_1, \ldots, \xi_k) \longrightarrow \pi_Y \circ \pi(\Phi(u \cdot \xi_1 \cdots \xi_k)) \in Y$$

are called the kth infinitesimal and the kth exterior components of Φ, respectively.

For all $k > 1$ the infinitesimal and the exterior components of a $\mathbb{Z}_2$-graded coalgebra morphism are totally symmetric and even. By the universal property of the symmetric tensor product the infinitesimal Φ_k^+ and the exterior $\Phi_k^\wedge$ components uniquely extend to maps on $U \times S^k(X)$ and $U \times \wedge^k(X)$, respectively, which are linear and even in the second variable. For the sake of simplicity the same symbols will be used for the components and for their extensions above. Note that the infinitesimal Φ^+, and the exterior $\Phi^\wedge$ parts of Φ are uniquely determined by their components, $\{\Phi_k^+\}_{k\geq 1}$, and $\{\Phi_k^\wedge\}_{k\geq 1}$, respectively.

The following proposition is a consequence of the universal property of the symmetric algebra with respect to the coalgebra morphisms (Theorem 2.1.2). It asserts that a morphism $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ of $\mathbb{Z}_2$-graded coalgebras is uniquely determined by its underlying and infinitesimal parts.

Proposition 3.2.1. *Let X, Y be $\mathbb{Z}_2$-graded spaces and $\mathcal{D}_X(U), \mathcal{D}_Y(V)$ subcoalgebras over $U \subset X_0$ and $V \subset Y_0$, respectively. Let $\phi^+ : U \times S(X)^+ \to Y$ be a morphism of $\mathbb{Z}_2$-graded spaces in the second variable, and $\phi : U \to V$ an arbitrary map.*

Then there exists a unique morphism $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ of $\mathbb{Z}_2$-graded coalgebras such that $\Phi^0 = \phi$, and $\Phi^+ = \phi^+$.

Moreover, for every $u \in U, \alpha \in S(X)$

$$\Phi(u, \alpha) = \phi(u) \cdot * \exp \phi_u^+(\alpha), \tag{14}$$

where

$$\phi_u^+ \equiv \phi^+(u,.) : S(X)^+ \longrightarrow Y,$$
$$*\exp\phi_u^+(\alpha) \equiv \sum_{k\geq 0} \frac{1}{k!}\phi_u^{+k}(\alpha),$$
$$\phi_u^{+0}(\alpha) \equiv u_{S(Y)} \circ \varepsilon_{S(X)}(\alpha),$$
$$\phi_u^{+k}(\alpha) \equiv \underbrace{\phi_u^+ \circ \pi^+ * \cdots * \phi_u^+ \circ \pi^+(\alpha)}_{k}, \quad k \geq 1.$$

$\pi^+ : S(X) \to S(X)^+$ *denotes the canonical projection, and* $*$ *is the convolution in* $\mathrm{Hom}(\mathcal{D}_{X_u}, S(X))$.

It follows that a $\mathbb{Z}_2$-graded coalgebra morphism $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ is uniquely determined by the underlying map Φ^0 and the infinitesimal components $\{\Phi_k^+\}_{k\geq 1}$. In the following definition we introduce the notion of smooth coalgebra morphism by imposing some additional requirements on the maps Φ^0, $\{\Phi_k^+\}_{k\geq 1}$.

Definition 3.2.2. Let X, Y be $\mathbb{Z}_2$-graded Fréchet spaces and $\mathcal{D}_X(U)$, $\mathcal{D}_Y(V)$ open subcoalgebras of $\mathcal{D}_X$ and $\mathcal{D}_Y$, respectively.

A morphism $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ of $\mathbb{Z}_2$-graded coalgebras is said to be smooth if the following conditions are stasified:

1. The underlying part $\Phi^0 : U \to V$ is a continuous map and for all $x \in X_0$, $u \in U$, the directional derivative $D^1\Phi^0(u; x)$ exists.
2. For every $k \geq 1$, the kth infinitesimal component

$$\Phi_k^+ : U \times \underbrace{X \times \cdots \times X}_{k} \longrightarrow Y$$

is jointly continuous with respect to the cartesian product topology on $U \times X^{\times k}$ and the directional derivatives $D^1\Phi_k^+(u, a_1, \ldots, a_k; x)$ with respect to the first variable exist for all $x \in X_0$, $u \in U$, $a_i \in X$.

3. For every $u \in U$, $x \in X_0$, $a_i \in X$ the following relations hold:

$$D^1\Phi^0(u; x) = \Phi_1^+(u, x), \tag{15}$$
$$D^1\Phi_k^+(u, a_1, \ldots, a_k; x) = \Phi_{k+1}^+(u, a_1, \ldots, a_k, x). \tag{16}$$

Proposition 3.2.2. *Let* $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ *be a smooth morphism of* $\mathbb{Z}_2$*-graded coalgebras and* $\{x_j\}_{j=1}^l$ *an arbitrary sequence of elements of* X_0.

1. *The underlying part* $\Phi^0 : U \to V$ *of* Φ *is a smooth map and for every* $u \in U$ *the* l*th order partial derivatives satisfy the relation*

$$D^l\Phi^0(u; x_1, \ldots, x_l) = \Phi_l^+(u, x_1, \ldots, x_l). \tag{17}$$

2. *For every $k \geq 1$ the kth exterior component $\Phi_k^\wedge$ of Φ is a smooth map. For every $u \in U$, $\xi_i \in X_1$ the lth order partial derivatives with respect to the first variable are given by the formula*

$$D^l \Phi^\wedge(u, \xi_1, \ldots, \xi_k; x_1, \ldots, x_l) = \Phi^+_{k+l}(u, \xi_1, \ldots, \xi_k, x_1, \ldots, x_l). \tag{18}$$

3. *For every $k \leq 1$ the kth infinitesimal component ϕ_k^+ of Φ is a smooth map. For every $u \in U$, $a_i \in X$ the lth order partial derivatives satisfy the relation*

$$D^l \Phi_k^+(u, a_1, \ldots, a_k; x_1, \ldots, x_l) = \Phi^+_{k+l}(u, a_1, \ldots, a_k, x_1, \ldots, x_l). \tag{19}$$

Proof.
1. By condition 1 of Definition 3.2.2 the directional derivative $D^1\Phi^0(u; x)$ exists for all $u \in U$ and $x \in X_0$. By condition 3 one gets relation (17) for $l = 1$. Since by condition 2 the first component of Φ^+ is jointly continuous on $U \times X$ so is $D^1\Phi^0$ on $U \times X_0$, hence Φ^0 is C^1. The C^l smoothness and relation (17) for arbitrary l follow from induction on l.
2. Let us fix $k \geq 1$. Repeating the reasoning above one gets that $\Phi_k^\wedge$ is C^1 separately in the first variable and relation (18) holds for $l = 1$. But $\Phi_k^\wedge$ is linear in the second variable and by condition 2 of Definition 3.2.2, jointly continuous. It follows [16] that $\Phi_k^\wedge$ is jointly C^1. The induction on l yields the C^l-smoothness and relation (18) for arbitrary $x_i \in X_0$ and for all l.
3. It is a straightforward consequence of conditions 1 and 2. □

Let $\mathcal{D}_X(U), \mathcal{D}_Y(V)$ be open subcoalgebras. For $k \geq 1$ and $X_1 \neq \{0\}$ we introduce the space $C_k^\wedge(\mathcal{D}_X(U), \mathcal{D}_Y(V))$ of all smooth maps

$$\phi_k : U \times \underbrace{X_1 \times \cdots \times X_1}_{k} \longrightarrow Y,$$

which are k-linear, totally symmetric, and even in the second variable. Note that in the definition above X_1 is regarded as a purely odd graded Fréchet space $\{o\} \oplus X_1$. Thus the maps ϕ_k are totally antisymmetric in the usual sense. Let $C_0^\wedge(\mathcal{D}_X(U), \mathcal{D}_Y(V)) \equiv C^\infty(U, V)$ be the space of all smooth maps from the open subset $U \subset X_0$ to $V \subset Y_0$ and

$$C^\wedge(\mathcal{D}_X(U), \mathcal{D}_Y(V)) \equiv \mathop{\times}_{k=0}^{\infty} C_k^\wedge(\mathcal{D}_X(U), \mathcal{D}_Y(V)).$$

For $X_1 = \{o\}$ we put $C^\wedge(\mathcal{D}_X(U), \mathcal{D}_Y(V)) \equiv C^\infty(U, V)$.

As a consequence of Propositions 3.2.1 and 3.2.2 one gets the following characterisation of the space $\mathrm{Mor}(\mathcal{D}_X(U), \mathcal{D}_Y(V))$ of all smooth morphisms $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ of open subcoalgebras

Theorem 3.2.1. *The map*

$$\mathrm{Mor}(\mathcal{D}_X(U), \mathcal{D}_Y(V)) \ni \Phi \longrightarrow (\Phi^0, \Phi_1^\wedge, \Phi_2^\wedge, \ldots) \in C^\wedge(\mathcal{D}_X(U), \mathcal{D}_Y(V))$$

is bijective.

In order to complete the construction of the model category we shall show that the composition of smooth coalgebra morphisms is smooth. For this purpose we start with the description of the composition of $\mathbb{Z}_2$-graded coalgebra morphisms in terms of components.

Proposition 3.2.3. *Let $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$, $\Psi : \mathcal{D}_Y(V) \to \mathcal{D}_Z(W)$ be morphisms of $\mathbb{Z}_2$-graded coalgebras. The underlying part $(\Psi \circ \Phi)^0$ and the kth infinitesimal components $(\Psi \circ \Phi)_k^+$ of the $\mathbb{Z}_2$-graded coalgebra morphism $\Psi \circ \Phi : \mathcal{D}_X(U) \to \mathcal{D}_Z(W)$ are given by*

$$(\Psi \circ \Phi)^0 = \Psi^0 \circ \Phi^0,$$

and

$$(\Psi \circ \Phi)_k^+(u, a_1, \dots, a_k) = \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{\{P_1,\dots,P_i\} \\ |P_i|>0}} \sigma(\mathcal{X}, \mathcal{P}) \times \Psi_i^+(\Phi^0(u), \Phi_{|P_1|}^+(u, a_{P_1}), \dots, \Phi_{|P_i|}^+(u, a_{P_i})), \tag{20}$$

where the sum is over all nonempty partitions $\mathcal{P} = \{P_1, \dots, P_i\}$ of the index set $\{1, \dots, k\}$ and the notation of Remark 2.1.2 *is used.*

Proof. According to Definition 3.2.1 the kth infinitesimal component of $\Psi \circ \Phi$ is given by

$$(\Psi \circ \Phi)_k^+(u, a_1, \dots, a_k) = \pi_Y \circ \pi \circ \Psi \circ \Phi(u \cdot a_1 \cdots a_k).$$

Representing Φ in the formula above in terms of its infinitesimal components (Proposition 3.2.1) and using explicit form of comultiplication in $S(X)$ given in Proposition 2.1.3 one gets the result required. □

Theorem 3.2.2. *Let X, Y, Z be $\mathbb{Z}_2$-graded Fréchet spaces and $\mathcal{D}_X(U), \mathcal{D}_Y(V), \mathcal{D}_Z(W)$ open subcoalgebras of $\mathcal{D}_X$, $\mathcal{D}_Y$ and $\mathcal{D}_Z$, respectively*

If $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$, $\Psi : \mathcal{D}_Y(V) \to \mathcal{D}_Z(W)$ are smooth morphisms of $\mathbb{Z}_2$-graded coalgebras so is their composition $\Psi \circ \Phi : \mathcal{D}_X(U) \to \mathcal{D}_Z(W)$.

The underlying part of $\Psi \circ \Phi$ is the composition of underlying parts $(\Psi \circ \Phi)^0 = \Psi^0 \circ \Phi^0$. The kth exterior component of $\Psi \circ \Phi$ is given by

$$(\Psi \circ \Phi)_k^\wedge(u, \xi_1, \dots, \xi_k) = \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{\{P_1,\dots,P_i\} \\ |P_i|>0}} \sigma(\mathcal{X}, \mathcal{P}) \times \Psi_i^+(\Phi^0(u), \Phi_{|P_1|}^\wedge(u, \xi_{P_1}), \dots, \Phi_{|P_i|}^\wedge(u, \xi_{P_i})), \tag{21}$$

where the sum is over all nonempty partitions $\{P_1, \dots, P_i\}$ of the index set $\{1, \dots, k\}$.

Proof. By Proposition 3.2.2 Ψ^0, Φ^0 are smooth maps so is their composition $(\Psi \circ \Phi)^0 = \Psi^0 \circ \Phi^0$. By Proposition 3.2.3 the kth infinitesimal component $(\Psi \circ \Phi)_k^+$ can be expressed as a finite sum of compositions $\Psi_i^+ \circ (\Phi_{|P_1|}^+ \times \cdots \times \Phi_{|P_i|}^+) \circ \tilde{\mathcal{P}}$, where

$$\tilde{\mathcal{P}} : U \times X^{\times k} \ni (u, a_1, \dots, a_k) \longrightarrow (u, a_{|P_1|}) \times \cdots \times (u, a_{|P_i|}) \in \mathop{\times}_{j=1}^{i} (U \times X^{\times |P_j|}).$$

Since $\tilde{\mathcal{P}}$ is smooth with respect to the cartesian pproduct topology the smoothness of $(\Psi \circ \Phi)^+_k$ follows from the smoothness of Ψ^+_i and Φ^+_j.

It remains to check condition 3 of Definition 3.2.2. Using relation (15) for Ψ^0 and Φ^0 and (20) for $k = 1$ one gets

$$\begin{aligned} D^1(\Psi \circ \Phi)^0(u; x) &= D^1\Psi^0(\Phi^0(u); D^1\Phi^0(u; x)) \\ &= \Psi^+_1(\Phi^0(u), \Phi^+_1(u; x)) = (\Psi \circ \Phi)^+_1(u, x), \end{aligned}$$

for all $u \in U, x \in X_0$. Hence relation (15) of Definition 3.2.2 is satisfied.

Differentiating expression (20) for the kth infinitesimal component $(\Psi \circ \Phi)^+_k$ and using relations (15) and (16) for components of Ψ and Φ one has

$$\begin{aligned} & D^1(\Psi \circ \Phi)^+_k(u, a_1, \dots, a_k; x) \\ &= \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{\{P_1,\dots,P_i\} \\ |P_i|>0}} \sigma(\mathcal{X}, \mathcal{P}) \\ &\quad \times \Bigg[D^1\Psi^+_i(\Phi^0(u), \Phi^+_{|P_1|}(u, a_{P_1}), \dots, \Phi^+_{|P_i|}(u, a_{P_i}); D^1\Phi^0(u; x)) \\ &\quad + \sum_{j=1}^{i} \Psi^+_i(\Phi^0(u), \Phi^+_{|P_1|}(u, a_{P_1}), \dots, D^1\Phi^+_{|P_j|}(u, a_{P_j}; x), \dots, \Phi^+_{|P_i|}(u, a_{P_i})) \Bigg] \\ &= \sum_{i=2}^{k+1} \frac{1}{(i-1)!} \sum_{\substack{\{P_1,\dots,P_{i-1}\} \\ |P_i|>0}} \sigma(\mathcal{X}, \mathcal{P}) \\ &\quad \times \Psi^+_i(\Phi^0(u), \Phi^+_{|P_i|}(u, a_{P_1}), \dots, \Phi^+_{|P_i|}(u, a_{P_i}), \Phi^+_1(u, x)) \\ &\quad + \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{\{P_1,\dots,P_i\} \\ |P_i|>0}} \sigma(\mathcal{X}, \mathcal{P}) \\ &\quad \times \sum_{j=1}^{i} \Psi^+_i(\Phi^0(u), \Phi^+_{|P_1|}(u, a_{P_1}), \dots, \Phi^+_{|P_j|+1}(u, a_{P_j}, x), \dots, \Phi^+_{|P_i|}(u, a_{P_i})) \\ &= \sum_{i=1}^{k+1} \frac{1}{i!} \sum_{\substack{\{Q_1,\dots,Q_i\} \\ |Q_i|>0}} \sigma(\mathcal{X}', \mathcal{Q}) \Psi^+_i(\Phi^0(u), \Phi^+_{|Q_i|}(u, a_{Q_1}), \dots, \Phi^+_{|Q_i|}(u, a_{Q_i})) \\ &= (\Psi \circ \Phi)^+_{k+1}(u, a_1, \dots, a_k, x), \end{aligned}$$

where $\mathcal{P} = \{P_1, \ldots, P_i\}$ and $\mathcal{Q} = \{Q_i, \ldots, Q_i\}$ are partitions of the index sets $\{1, \ldots, k\}$ and $\{1, \ldots, k+1\}$, respectively, and $\mathcal{X}' = \{a_1, \ldots, a_{k+1}\}$, $a_{k+1} = x$.

Formula (21) is a special case of (20). □

3.3. Direct product

The considerations of Sections 3.1 and 3.2 leads to the following definition of the model category.

Definition 3.3.1. The objects of the model category **sc** of smooth graded coalgebras are open subcoalgebras $\mathcal{D}_X(U)$ where X runs over the category of graded Fréchet spaces and U over all open subsets of the even part X_0 of a graded Fréchet space X.

For any two objects $\mathcal{D}_X(U), \mathcal{D}_Y(V) \in$ **Osc** the space of morphisms **Msc**$(\mathcal{D}_X(U), \mathcal{D}_Y(V))$ consists of all smooth coalgebra morphisms $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$.

The composition of morphisms in **sc** is defined as a composition of coalgebra maps.

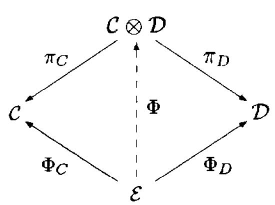
An isomorphism in the category **sc** is called a diffeomorphism of open subcoalgebras.

We shall show that in the model category defined above the direct product exists. We start with the corresponding result for the category of $\mathbb{Z}_2$-graded cocommutative coalgebras [40].

Theorem 3.3.1. *Let $(\mathcal{C}, \Delta_C, \varepsilon_C)$, $(\mathcal{D}, \Delta_D, \varepsilon_D)$ be $\mathbb{Z}_2$-graded cocommutative coalgebras. Define the maps*

$$\pi_C : \mathcal{C} \otimes \mathcal{D} \ni c \otimes d \longrightarrow \varepsilon_D(d) \cdot c \in \mathcal{C},$$
$$\pi_D : \mathcal{C} \otimes \mathcal{D} \ni c \otimes d \longrightarrow \varepsilon_C(c) \cdot d \in \mathcal{D}.$$

Then π_C, π_D are $\mathbb{Z}_2$-graded coalgebra morphisms and for every $\mathbb{Z}_2$-graded cocommutative coalgebra $(\mathcal{E}, \Delta_E, \varepsilon_D)$ and morphisms of $\mathbb{Z}_2$-graded coalgebras $\Phi_C : \mathcal{E} \to \mathcal{C}$, and $\Phi_D : \mathcal{E} \to \mathcal{D}$, there exists a unique morphism of $\mathbb{Z}_2$-graded coalgebras $\Phi : \mathcal{E} \to \mathcal{C} \otimes \mathcal{D}$ making the diagram



(22)

commute. The $\mathbb{Z}_2$-graded coalgebra morphism Φ is given by $\Phi = (\Phi_C \otimes \Phi_D) \circ \Delta_E$.

Definition 3.3.2. Let $\mathcal{D}_X(U), \mathcal{D}_Y(V) \in$ **Osc**. The tensor product $\mathcal{D}_X(U) \otimes \mathcal{D}_Y(V) \in$ **Osc** is defined as follows:

1. With respect to the coalgebra structure $\mathcal{D}_X(U) \otimes \mathcal{D}_Y(V)$ is the tensor product of graded cocommutative coalgebras.

2. With respect to the topological structure $\mathcal{D}_X(U) \otimes \mathcal{D}_Y(V)$ is identified with an open subcoalgebra of $\mathcal{D}_{X\oplus Y}$ by the canonical isomorphism

$$\mathcal{D}_X(U) \otimes \mathcal{D}_Y(V) \longrightarrow \mathcal{D}_{X\oplus Y}(U \times V)$$

where the direct sum topology on $X \oplus Y = (X_0 \oplus Y_0) \oplus (X_1 \oplus Y_1)$ is assumed.

Calculating infinitesimal components and checking the conditions of Definition 3.2.2 by explicit calculation of directional derivatives one gets the following:

Proposition 3.3.1.
1. *Let $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$, $\Phi' : \mathcal{D}_{X'}(U') \to \mathcal{D}_{Y'}(V')$ be smooth morphisms of open subcoalgebras. Then the tensor product of $\mathbb{Z}_2$-graded coalgebra morphisms*

$$\begin{aligned}&\Phi \otimes \Phi' : \mathcal{D}_X(U) \otimes \mathcal{D}_{X'}(U') \ni m \otimes m' \\ &\quad \to \Phi(m) \otimes \Phi'(m') \in \mathcal{D}_Y(V) \otimes \mathcal{D}_{Y'}(V')\end{aligned}$$

is smooth.

2. *Let ε_U, ε_V be counits of open subcoalgebras $\mathcal{D}_X(U)$, and $\mathcal{D}_Y(V)$, respectively. Then the maps*

$$\begin{aligned}P_U &\equiv I \otimes \varepsilon_V : \mathcal{D}_X(U) \otimes \mathcal{D}_Y(V) \longrightarrow \mathcal{D}_X(U),\\ P_V &\equiv \varepsilon_U \otimes I : \mathcal{D}_X(U) \otimes \mathcal{D}_Y(V) \longrightarrow \mathcal{D}_Y(V)\end{aligned}$$

are smooth morphisms of $\mathbb{Z}_2$-graded coalgebras.

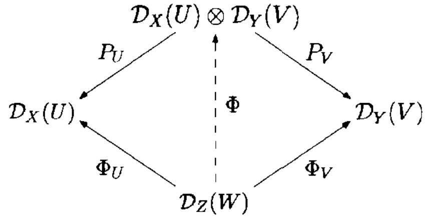
3. *The comultiplication $\Delta : \mathcal{D}_X(U) \to \mathcal{D}_X(U) \otimes \mathcal{D}_X(U)$ of an open subcoalgebra $\mathcal{D}_X(U)$ is a smooth morhism of $\mathbb{Z}_2$-graded coalgebras.*

Remark 3.3.1. The underlying parts and the exterior components of smooth $\mathbb{Z}_2$-graded coalgebra morphisms of the proposition above are given by

$$\begin{aligned}&(\Phi \otimes \Phi')^0(u, u') = (\Phi^0(u), \Phi'^0(u')),\\ &(\Phi \otimes \Phi')^\wedge_k((u, u'), \theta_1 \oplus \theta'_1, \dots, \theta_k \oplus \theta'_k)\\ &\quad = \Phi^\wedge_k(u, \theta_1, \dots, \theta_k) \oplus \Phi'^\wedge_k(u', \theta'_1, \dots, \theta'_k);\\ &(P_U)^0(u, v) = u,\\ &(P_U)^\wedge_1((u, v), \theta \oplus \eta) = \theta,\\ &(P_U)^\wedge_k((u, v), \theta_1 \oplus \eta_1, \dots, \theta_k \oplus \eta_k) = 0, \quad k \geq 2;\\ &\Delta^0(u) = (u, u),\\ &\Delta^\wedge_1(u, \theta) = \theta \oplus \theta,\\ &\Delta^\wedge_k(u, \theta_1, \dots, \theta_k) = 0, \quad k \geq 2.\end{aligned}$$

Theorem 3.3.2. *$(\mathcal{D}_X(U) \otimes \mathcal{D}_Y(V), P_U, P_V)$ is the direct product in the model category of smooth $\mathbb{Z}_2$-graded coalgebras.*

Proof. By Proposition 3.3.1 P_U, P_V are **sc**-morphisms. By Theorem 3.3.1 for any pair of smooth morphisms of $\mathbb{Z}_2$-graded coalgebras $\Phi_U : \mathcal{D}_Z(W) \to \mathcal{D}_X(U)$, and $\Phi_V : \mathcal{D}_Z(W) \to \mathcal{D}_Y(V)$, the map $\Phi = \Phi_U \otimes \Phi_V \circ \Delta_W$ is the unique $\mathbb{Z}_2$-graded coalgebra morphism for which the diagram



commutes. By Proposition 3.3.1 Φ is a composition of smooth morphisms of $\mathbb{Z}_2$-graded coalgebras. Hence Φ is smooth by Theorem 3.2.2. □

Remark 3.3.2. The underlying part and the exterior components of the smooth morphism $\Phi = \Phi_U \times \Phi_V \circ \Delta_W$ are given by

$$\Phi^0(w) = (\Phi_U^0(w), \Phi_V^0(w)),$$
$$\Phi_k^\wedge(w, \theta_1, \ldots, \theta_k) = (\Phi_U)_k^\wedge(w, \theta_1, \ldots, \theta_k) \oplus (\Phi_V)_k^\wedge(w, \theta_1, \ldots, \theta_k).$$

3.4. Subcategories $\mathbf{sc}_0$ *and* $\mathbf{sc}^<$

In this section we shall show that the model category **fm** of Fréchet manifolds and the model category **sm** of BLK supermanifolds are both full subcategories of the model category **sc** of smooth coalgebras. This justifies introduction of **sc** as an extension of **fm** and **sm**.

We define the category $\mathbf{sc}_0$ of even open coalgebras as the full subcategory of **sc** consisting of all objects of the form $\mathcal{D}_{X_0 \oplus \{0\}}(U)$ and all **sc**-morphisms between them. Similarly, the category $\mathbf{sc}^<$ of finite-dimensional open coalgebras is defined as the full subcategory of **sc** consisting of all objects of the form $\mathcal{D}_{\mathbb{R}^m \oplus \mathbb{R}^n}(U)$ with arbitrary $m, n \geq 0$, and all **sc**-morphisms between them. For notational convenience we shall introduce simplified symbols $\mathcal{D}(U) \equiv \mathcal{D}_{X_0 \oplus \{o\}}(U)$ and $\mathcal{D}_{m,n}(U) \equiv \mathcal{D}_{\mathbb{R}^m \oplus \mathbb{R}^n}(U)$ for objects of $\mathbf{sc}_0$ and $\mathbf{sc}^<$, respectively. Note that by definition, for all $\mathcal{D}(U), \mathcal{D}(V) \in \mathbf{Osc}_0$ and $\mathcal{D}_{m,n}(U), \mathcal{D}_{m',n'}(V) \in \mathbf{Osc}^<$ one has

$$\mathbf{Msc}_0(\mathcal{D}(U), \mathcal{D}(V)) = \mathbf{Msc}(\mathcal{D}(U), \mathcal{D}(V)),$$
$$\mathbf{Msc}^<(\mathcal{D}_{m,n}(U), \mathcal{D}_{m',n'}(V)) = \mathbf{Msc}(\mathcal{D}_{m,n}(U), \mathcal{D}_{m',n'}(V)).$$

By Theorem 3.3.2 both $\mathbf{sc}_0$ and $\mathbf{sc}^<$ inherit the direct product from **sc**.

As a consequence of Theorem 3.2.1 and 3.2.2 one has the following.

Proposition 3.4.1. *The correspondence*

$$\mathbf{Osc}_0 \ni \mathcal{D}(U) \longrightarrow U \in \mathbf{Ofm},$$
$$\mathbf{Msc}_0(\mathcal{D}(U), \mathcal{D}(V)) \ni \Phi \longrightarrow \Phi^0 \in \mathbf{Mfm}(U, V) \tag{23}$$

is an equivalence of categories.

It follows that the model category **fm** of Fréchet manifolds can be identified with the full subcategory $\mathbf{sc}_0$ of the category **sc** of open coalgebras.

Remark 3.4.1. The inverse to the correspondence (23) is given by

$$\begin{aligned}&\mathbf{Ofm} \ni U \longrightarrow \mathcal{D}(U) \in \mathbf{Osc}_0,\\ &\mathbf{Mfm}(U, V) \ni \phi \longrightarrow \phi_* \in \mathbf{Msc}_0(\mathcal{D}(U), \mathcal{D}(V)),\end{aligned}$$

where ϕ_* is a unique smooth morphism of coalgebras such that $(\phi_*)^0 = \phi$. Using formula (1) of Proposition 2.1.3 for comultiplication in $S(X)$ and formulae (14) of Proposition 3.2.1 and (17) of Proposition 3.2.2 one gets for $u \in U, x_1, \dots, x_k \in X$,

$$\begin{aligned}&\phi_*(u \cdot x_i \cdots x_k)\\ &\quad = \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{\{P_1,\dots,P_i\}\\ |P_i|>0}} \phi(u) \cdot D^{|P_1|}\phi(u; x_{P_1}) \cdots D^{|P_i|}\phi(u; x_{P_i}),\end{aligned} \tag{24}$$

where the sum runs over all nonempty i-partitions of the index set $\{1, \dots, k\}$.

Let us now compare $\mathcal{D}(U)$ to the dual coalgebra (see Appendix A.5) $C^\infty(U)^\circ$ of the algebra $C^\infty(U)$ of smooth real-valued functions on an open subset U of a Fréchet space X. For this purpose we introduce the pairing

$$\langle .,. \rangle_U : \mathcal{D}(U) \times C^\infty(U) \longrightarrow \mathbb{R}$$

defined, for all $u \in U, x_1, \dots, x_k \in X$, and $f \in C^\infty(U)$ by

$$\begin{aligned}&\langle u, f \rangle_U \equiv f(u),\\ &\langle u \cdot x_1 \cdots x_k, f \rangle_U \equiv d^k f(u; x_1, \dots, x_k).\end{aligned} \tag{25}$$

Proposition 3.4.2. *Let U be an open subset of a Fréchet space X, $(\mathcal{D}(U), \Delta_U, \varepsilon_U)$ the even open coalgebra, and $C^\infty(U)$ the algebra of smooth functions on U.*

1. *For all $f, g \in C^\infty(U), \omega \in \mathcal{D}(U)$,*

$$\begin{aligned}&\langle \omega, f \cdot g \rangle_U = \sum_{(\omega)} \langle \omega_{(1)}, f \rangle_U \langle \omega_{(2)}, g \rangle_U,\\ &\langle \omega, 1 \rangle_U = \varepsilon_U(\omega),\end{aligned}$$

where $\Delta_U \omega = \sum_{(\omega)} \omega_{(1)} \otimes \omega_{(2)}$.

2. *Let $\phi : U \to V$ be a smooth map of open subsets of Fréchet spaces. Then*

$$\langle \phi_* \omega, f \rangle_V = \langle \omega, f \circ \phi \rangle_U,$$

for all $\omega \in \mathcal{D}(U)$, $f \in C^\infty(V)$.

The first part follows from the multiple Leibnitz rule (Proposition 2.2.2) and the explicit formula for the comultiplication in $S(X)$ (Proposition 2.1.3). The second part is a straightforward consequence of the multiple chain rule (Proposition 2.2.1) and formula (24) for ϕ_*.

Proposition 3.4.3. *With the notation of Proposition* 3.4.2 *the map*

$$\mathcal{D}(U) \ni \alpha \longrightarrow \langle \alpha, . \rangle \in \mathcal{C}^{\infty}(U)^{\circ} \tag{26}$$

is an injective morphism of $\mathbb{Z}_2$*-graded coalgebras.*

Proof. By Propositions 3.4.2 and 2.5.3 for all $\omega \in \mathcal{D}(U)$, $\langle \omega, . \rangle_U \in \mathcal{C}^{\infty}(U)^{\circ}$. By Definition 2.5.1 of dual coalgebra and Proposition 3.4.2 (26) is a morphism of $\mathbb{Z}_2$-graded coalgebras. Using the Hahn–Banach theorem for Fréchet spaces one can show that the pairing (25) is nonsingular which implies the injectivity of the map (26). □

Remark 3.4.2. The image of the map (26) is a subcoalgebra of $\mathcal{C}^{\infty}(U)^{\circ}$ consisting of all finite linear combinations of evaluations of directional derivatives of arbitrary order. This subcoalgebra will be called the *coalgebra of Dirac distributions on U*. For infinite-dimensional Fréchet spaces this is a proper subcoalgebra of $\mathcal{C}^{\infty}(U)^{\circ}$.

We shall proceed to the model category of BLK supermanifolds. Let $F = (F^0, F_{.}) : \mathcal{S}_U^{m,n} \to \mathcal{S}_V^{m',n'}$ be an **sm**-morphism and $\{F_k^{\wedge}\}_{k=1}^n$ its exterior components. The collection $\{F^0, F_1^{\wedge}, \ldots, F_n^{\wedge}\}$ can be regarded as a point in the space $\mathcal{C}^{\wedge}(\mathcal{D}_{m,n}(U), \mathcal{D}_{m',n'}(V))$. By Theorem 3.2.1 there exists a unique smooth $\mathbb{Z}_2$-graded coalgebra map with the underlying part F^0 and with the exterior components $\{F_k^{\wedge}\}_{k=1}^n$. We denote this map by F_*.

Let $G \to (G^0, G_{.}) : \mathcal{S}_V^{m',n'} \to \mathcal{S}_W^{m'',n''}$ be another **sm**-morphism. Then by the above construction one has a smooth $\mathbb{Z}_2$-graded coalgebra map $G_* : \mathcal{D}_{m',n'}(V) \to \mathcal{D}_{m'',n''}(W)$. Comparing formula (12) for the exterior components of composition of **sm**-morphisms (Remark 2.3.2) with the corresponding one (21) for the composition of smooth $\mathbb{Z}_2$-graded coalgebra morphisms (Theorem 3.2.2) one gets $(G \circ F)_* = G_* \circ F_*$.

Proposition 3.4.4. *The correspondence*

$$\begin{aligned} &\mathbf{Osm} \ni \mathcal{S}_U^{m,n} \longrightarrow \mathcal{D}_{m,n}(U) \in \mathbf{Osc}^{<}, \\ &\mathbf{Msm}(\mathcal{S}_U^{m,n}, \mathcal{S}_V^{m',n'}) \ni F = (F^0, F_{.}) \longrightarrow F_* \in \mathbf{Msc}^{<}(\mathcal{D}_{m,n}(U), \mathcal{D}_{m',n'}(V)) \end{aligned} \tag{27}$$

is an equivalence of cotegories.

Let $\mathcal{S}^{m,n}(U)$ be the $\mathbb{Z}_2$-graded algebra of superfunctions of a superdomain $\mathcal{S}_U^{m,n}$, and

$$\langle \cdot, \cdot \rangle_U : \mathcal{D}_{m,n}(U) \times \mathcal{S}^{m,n}(U) \longrightarrow \mathbb{R}$$

a pairing defined by

$$\begin{aligned} &\langle u, f \rangle_U \equiv f^0(u), \\ &\langle u \cdot a_1 \cdots a_k, f \rangle_U \equiv \tilde{D}^k f(u; a_1, \ldots, a_k) \end{aligned} \tag{28}$$

for all $u \in U, a_1, \ldots, a_k \in \mathbb{R}^m \oplus \mathbb{R}^n$, and $f \in \mathcal{S}^{m,n}(U)$.

As a consequence of the multiple Leibnitz rule (Proposition 2.3.3) and the multiple chain rule (Proposition 2.3.5) for superfunctions one has the following **sm**-counterpart of Proposition 3.4.2.

Proposition 3.4.5. *Let $\mathcal{S}^{m,n}(U)$ be the $\mathbb{Z}_2$-graded algebra of superfunctions of a superdomain $\mathcal{S}_U^{m,n}$, and $(\mathcal{D}_{m,n}(U), \Delta_U, \varepsilon_U)$ the open coalgebra $\mathcal{D}_{m,n}(U)$.*
1. *For all $f, g \in \mathcal{S}^{m,n}(U)_0 \cup \mathcal{S}^{m,n}(U)_1$, $\omega \in \mathcal{D}_{m,n}(U)$,*

$$\langle \omega, f \cdot g \rangle_U = \sum_{(\omega)} (-1)^{|f||\omega_{(2)}|} \langle \omega_{(1)}, f \rangle_U \langle \omega_{(2)}, g \rangle_U,$$
$$\langle \omega, 1 \rangle_U = \varepsilon_U(\omega),$$

where $\Delta_U \omega = \sum_{(\omega)} \omega_{(1)} \otimes \omega_{(2)}$.

2. *Let $F = (F^0, F_\cdot) : \mathcal{S}_U^{m,n} \to \mathcal{S}_V^{m',n'}$ be an* **sm**-*morphism. Then*

$$\langle F_* \omega, f \rangle_V = \langle \omega, F_V f \rangle_U$$

for all $\omega \in \mathcal{D}_{m,n}(U)$, $f \in \mathcal{S}^{m',n'}(V)$.

Proposition 3.4.6. *With the notation of Proposition* 3.4.5 *the map*

$$\mathcal{D}_{m,n}(U) \ni \omega \longrightarrow \langle \omega, . \rangle \in \mathcal{S}^{m,n}(U)^\circ \tag{29}$$

is an isomorphism of $\mathbb{Z}_2$-graded coalgebras.

Proof. Since the pairing (28) is nonsingular the map (29) is injective. By Proposition 3.4.5 it is a morphism of $\mathbb{Z}_2$-graded coalgebras. In particular it preserves the coradical filtration

$$\mathcal{D}_{m,n}(U)_u = \bigcup_{k \geq 0} \mathcal{D}_{m,n}(U)_u^{(k)}$$

of the irreducible components $\mathcal{D}_{m,n}(U)_u$, $u \in U$. Hence, for all $u \in U$, $k \geq 0$ one has the injective maps

$$\mathcal{D}_{m,n}(U)_u^{(k)} \ni \alpha \longrightarrow \langle \alpha, . \rangle \in \mathcal{S}^{m,n}(U)_u^{\circ(k)}. \tag{30}$$

By Proposition A.5.6 $\mathcal{S}^{m,n}(U)_u^{\circ(k)} = (\mathcal{S}^{m,n}(U)/I_u^{k+1})'$ and therefore

$$\dim(\mathcal{D}_{m,n}(U)_u^{(k)}) = \dim(\mathcal{S}^{m,n}(U)_u^{\circ(k)}) < +\infty.$$

It follows that the maps (30) are surjective for all $k \geq 0$, and $u \in U$, and so is the map (29). □

Remark 3.4.3. By Proposition 3.4.6 the open coalgebra $\mathcal{D}_{m,n}(U)$ can be identified with the dual coalgebra of the algebra of superfunctions $\mathcal{S}^{m,n}(U)$. The *coalgebra of Dirac distributions on the superdomain $\mathcal{S}_U^{m,n}$* is defined as coalgebra of all finite linear combinations of evaluations of differentiations of superfunctions. This coalgebra coincides with the dual coalgebra $\mathcal{S}^{m,n}(U)^\circ$ [19].

3.5. Superfunctions

Propositions 3.4.1 and 3.4.3 of Section 3.4 provide characterisation of even open coalgebras as objects dual to algebras of smooth functions on open subsets of Fréchet spaces. Similarly by Propositions 3.4.4 and 3.4.6 finite-dimensional open coalgebras are dual coalgebras to algebras of superfunctions on superdomains. Using this duality smooth functions in both cases can be regarded as linear functionals on corresponding coalgebras. This leads to the following definition of superfunction in general case.

Definition 3.5.1. Let $\mathcal{D}_X(U)$ be an open coalgebra. A linear functional $f \in \mathcal{D}_X(U)'$ is called a superfunction on $\mathcal{D}_X(U)$ if the following conditions are staisfied:
1. For each $k \geq 0$ the function

$$f_k : U \times \underbrace{X \times \cdots \times X}_{k} \ni (u, a_1, \ldots, a_k) \longrightarrow \langle f, u \cdot a_1 \cdots a_k \rangle \in \mathbb{R}$$

is jointly continuous with respect to the cartesian product topology on $U \times X^{\times k}$.
2. For each $k \geq 0$ and for all $x \in X_0$, $u \in U$, and $a_1, \ldots, a_k \in X$ the partial derivative

$$D_x \langle f, u \cdot a_1 \cdots a_k \rangle = \lim_{\epsilon \to 0} \frac{\langle f, (u + \epsilon x) \cdot a_1 \cdots a_k \rangle - \langle f, u \cdot a_1 \cdots a_k \rangle}{\epsilon}$$

exists.
3. For each $k \geq 0$ and for all $x \in X_0$, $u \in U$, and $a_1, \ldots, a_k \in X$

$$D_x \langle f, u \cdot a_1 \cdots a_k \rangle = \langle f, u \cdot a_1 \cdots a_k \cdot x \rangle.$$

Remark 3.5.1. Let $\mathcal{D}_X(U)'$ be the full algebraic dual of an open coalgebra $\mathcal{D}_X(U)$. Let us consider the inclusion of $\mathbb{Z}_2$-graded spaces

$$i : \mathcal{D}_X(U)' \otimes \mathcal{D}_X(U)' \longrightarrow (\mathcal{D}_X(U) \otimes \mathcal{D}_X(U))'$$

defined for each $f, g \in \mathcal{D}_X(U)'_0 \cup \mathcal{D}_X(U)'_1$, and $\alpha, \beta, \in \mathcal{D}_X(U)_0 \cup \mathcal{D}_X(U)_1$ by

$$\langle i(f \otimes g), \alpha \otimes \beta \rangle = (-1)^{|g||\alpha|} \langle f, \alpha \rangle \langle g, \beta \rangle.$$

$\mathcal{D}_X(U)'$ has the structure of $\mathbb{Z}_2$-graded commutative algebra with multiplication

$$M : \mathcal{D}_X(U)' \otimes \mathcal{D}_X(U)' \xrightarrow{i} (\mathcal{D}_X(U) \otimes \mathcal{D}_X(U))' \xrightarrow{\Delta'} \mathcal{D}_X(U)',$$

and unit $u : \mathbb{R} \xrightarrow{\varepsilon'} \mathcal{D}_X(U)'$.

The explicit formula for the product $f \cdot g$ reads

$$\langle f \cdot g, u \cdot a_1 \cdots a_k \rangle = \sum_{\mathcal{P} = \{P_1, P_2\}} \sigma(\mathcal{X}, \mathcal{P}) (-1)^{|g||a_{P_1}|} \langle f, u \cdot a_{P_1} \rangle \langle g, u \cdot a_{P_2} \rangle, \quad (31)$$

where $u \in U, a_1, \ldots, a_k \in X$, the sum runs over all two-partitions of the index set $\{1, \ldots, k\}$, and the notation of Remark 2.1.2 is used. The unit $\mathbf{1} = u(1)$ is given by

$$\langle \mathbf{1}, u \rangle = 1, \quad \langle \mathbf{1}, u \cdot a_1 \cdots a_k \rangle = 0$$

for all $u \in U, a_1, \ldots, a_k \in X$.

Differentiating the formulae above with respect to the variable u and checking the conditions of Definition 3.5.1 one gets the following.

Proposition 3.5.1. *The subspace $\mathcal{D}_X(U)^\wedge \subset \mathcal{D}_X(U)'$ consisting of all superfunctions on an open coalgebra $\mathcal{D}_X(U)$ is a $\mathbb{Z}_2$-graded subalgebra of $\mathcal{D}_X(U)'$.*

Definition 3.5.2. Let f be a superfunction on an open coalgebra $\mathcal{D}_X(U)$. The map

$$f^0 : U \ni u \longrightarrow \langle f, u \rangle \in \mathbb{R}$$

is called the underlying part of the superfunction f. For each $k \geq 1$ the kth exterior component $f_k^\wedge$ of the superfunction f is defined as a map

$$f_k^\wedge : U \times \underbrace{X_1 \times \cdots \times X_1}_{k} \ni (u, \xi_1 \ldots, \xi_k) \longrightarrow \langle f, u \cdot \xi_1 \cdots \xi_k \rangle \in \mathbb{R}.$$

It follows from Definition 3.5.1 that the underlying part and the exterior components of a superfunction f on an open coalgebra $\mathcal{D}_X(U)$ are smooth functions on U and $U \times X_1^{\times k}$, respectively. For all $u \in U; x_1, \ldots, x_k \in X_0; \xi_1, \ldots, \xi_l \in X_1$ one has

$$\langle f, u \cdot \xi_1 \cdots \xi_l \cdot x_k \cdots x_1 \rangle = D_{x_1} \cdots D_{x_k} \langle f, u \cdot \xi_1 \cdots \xi_l \rangle,$$

where D_x denotes the partial derivative with respect to u-variable in the direction $x \in X_0$. This implies that the superfunction is uniquely determined by f^0, and $\{f_k^\wedge\}_{k \geq 1}$.

Let $\mathcal{D}_X(U)$ be an open coalgebra and

$$R : \mathcal{D}_X(U) \times S(X) \ni (\omega, u \cdot \alpha) \longrightarrow u \cdot \alpha \cdot \omega \in \mathcal{D}_X(U)$$

the right action of the Hopf algebra $S(X)$ on $\mathcal{D}_X(U)$ introduced in Section 3.1. For each $\omega \in S(X)$ one has a linear map $R'_\omega : \mathcal{D}_X(U)' \longrightarrow \mathcal{D}_X(U)'$ given by

$$\langle R'_\omega f, u \cdot \alpha \rangle = \langle f, R_\omega(u \cdot \alpha) \rangle = \langle f, u \cdot \alpha \cdot \omega \rangle$$

for all $u \in U, \alpha \in S(X)$. One can easily check that if $f \in \mathcal{D}_X(U)'$ is a superfunction so is $R'_\omega f$.

Definition 3.5.3. Let f be a superfunction on an open coalgebra $\mathcal{D}_X(U)$. For each $\omega \in S(X)$ the superfunction $D_\omega f = R'_\omega f$ is called the derivative of f in the direction ω.

The following proposition says that the notions of superfunction on open coalgebra (Definition 3.5.1) and its derivative (Definition.3.5.3) are generalisations of the corresponding notions both in the smooth Fréchet geometry and in the finite-dimensional BLK supergeometry.

Proposition 3.5.2.

1. *Let $X = X_0 \oplus \{0\}$ be a purely even $\mathbb{Z}_2$-graded Fréchet space. Let U be an open subset of X_0 and let*

$$\langle\cdot,\cdot\rangle_U : \mathcal{D}(U) \times \mathcal{C}^\infty(U) \longrightarrow \mathbb{R}$$

be the pairing of Proposition 3.4.2. *Then the map*

$$\mathcal{C}^\infty(U) \ni f \longrightarrow \bar{f} = \langle\cdot, f\rangle_U \in \mathcal{D}_X(U)^\wedge$$

is an isomorphism of $\mathbb{Z}_2$-graded algebras. Moreover, for all $u \in U, x_1, \ldots, x_k \in X$ one has

$$D_{x_1\ldots x_k}\overline{f}(u) = \overline{D^k f}(u, x_1, \ldots, x_k).$$

2. *Let $X = \mathbb{R}^m \oplus \mathbb{R}^n$ be a finite-dimensional $\mathbb{Z}_2$-graded Fréchet space. Let U be an open subset of $\mathbb{R}^m$ and*

$$\langle\cdot,\cdot\rangle_U : \mathcal{D}_{m,n}(U) \times \mathcal{S}^{m,n}(U) \longrightarrow \mathbb{R}.$$

the pairing of Proposition 3.4.5. *Then the map*

$$\mathcal{S}^{m,n}(U) \ni f \longrightarrow \overline{f} = \langle\cdot, f\rangle_U \in \mathcal{D}_X(U)^\wedge$$

is an isomorphism of $\mathbb{Z}_2$-graded algebras. Moreover, for all $u \in U, a_1, \ldots, a_k \in \mathbb{R}^m \oplus \mathbb{R}^n$ one has

$$D_{a_1\cdots a_k}\overline{f}(u,\theta) = \overline{D^k f}(u,\theta; a_1, \ldots, a_k).$$

Remark 3.5.2. Let $\Phi : D_X(U) \longrightarrow \mathcal{D}_Y(V)$ be a smooth morphism of open coalgebras. Then the dual map $\Phi' : \mathcal{D}_Y(V)' \longrightarrow \mathcal{D}_X(U)'$ is a morphism of $\mathbb{Z}_2$-graded algebras. Let f be a superfunction on $\mathcal{D}_Y(V)$. For all $u \in U, a_1, \ldots, a_k \in X$ one has

$$\begin{aligned}\langle\Phi' f, u\cdot a_1\cdots a_k\rangle &= \langle f, \Phi(u\cdot a_1\cdots a_k)\rangle \\ &= \sum_{i=1}^{k}\frac{1}{i!}\sum_{\substack{\{P_1,\ldots,P_i\}\\ |P_i|>0}} \sigma(\mathcal{X},\mathcal{P})\langle f, \Phi^0(u)\cdot\Phi^+_{|P_1|}(u, a_{P_1})\cdots\Phi^+_{|P_i|}(u, a_{P_i})\rangle.\end{aligned} \tag{32}$$

where the sum runs over all nonempty partitions of the index set $\{1, \ldots, k\}$. Following the proof of Theorem 3.2.2 one can show that the functional $\Phi' f$ is a superfunction on $\mathcal{D}_X(U)$. It follows that Φ' defines a morphism of $\mathbb{Z}_2$-graded algebras

$$\Phi^+ : \mathcal{D}_Y(V)^\wedge \ni f \longrightarrow \Phi' f \in \mathcal{D}_X(U)^\wedge$$

The superfunction $\Phi^+ f = \Phi' f$ is called the *pull-back* of the superfunction f.

Differentiating formulae (31) and (32) according to Definition 3.5.3 one gets the multiple Leibnitz and the chain rules for superfunctions.

Proposition 3.5.3. *Let $\mathcal{D}_X(U)^\wedge$, $\mathcal{D}_Y(V)^\wedge$ be algebras of superfunctions on open coalgebras $\mathcal{D}_X(U)$, and $\mathcal{D}_Y(V)$, respectively. Let $\Phi : \mathcal{D}_X(U) \longrightarrow \mathcal{D}_Y(V)$ be a smooth morphisms of open coalgebras.*

1. *For each $f, g \in \mathcal{D}_X(U)^\wedge$ and $a_1, \dots, a_k \in X$*

$$D_{a_1 \cdots a_k}(f \cdot g) = \sum_{\mathcal{P}=\{P_1,P_2\}} \sigma(\mathcal{X}, \mathcal{P})(-1)^{|f||a_{P_2}|} D_{a_{P_1}} f \cdot D_{a_{P_2}} g, \tag{33}$$

where the sum runs over all two-partitions of the index set $\{1, \dots k\}$ and the convention $D_{a_\emptyset} f = D_1 f = f$ is used.

2. *For each $g \in \mathcal{D}_Y(V)^\wedge$ and $a_1, \dots, a_k \in X$*

$$\begin{aligned} &(D_{a_1 \cdots a_k} \Phi^* g)^0(u) \\ &= \sum_{l=1}^{k} \frac{1}{l!} \sum_{\substack{\{P_1,\dots,P_l\} \\ P_i \neq \emptyset}} \sigma(\mathcal{X}, \mathcal{P}) \langle g, \Phi^0(u) \cdot \Phi^+(u, a_{P_1}) \cdots \Phi^+(u, a_{P_l}) \rangle, \end{aligned} \tag{34}$$

where the sum runs over all nonempty partitions of the index set $\{1, \dots, k\}$.

Let $\mathcal{D}_X(U)$ be an open coalgebra. For each pair of open subsets $U'' \subset U' \subset U$ we define the restriction map

$$\varrho_{U'U''} : \mathcal{D}_X(U')^\wedge \longrightarrow \mathcal{D}_X(U'')^\wedge$$

as dual to the inclusion $\mathcal{D}_X(U'') \subset \mathcal{D}_X(U')$. The assignment for each open subset $U' \subset U$ the $\mathbb{Z}_2$-graded algebra $\mathcal{D}_X(U')^\wedge$ of superfunctions with the restriction maps above defines a sheaf $\mathcal{D}_U^\wedge = (U, \mathcal{D}_X(.)^\wedge)$ of $\mathbb{Z}_2$-graded algebras.

Let $\Phi : \mathcal{D}_X(U) \to \mathcal{D}_Y(V)$ be a smooth morphism of open coalgebras. For each open $V' \subset V$ the restriction

$$\Phi_{V'} : \mathcal{D}_X(\Phi^{0^{-1}}(V')) \ni \mu \longrightarrow \Phi(\mu) \in \mathcal{D}_Y(V')$$

is a smooth morphism of open coalgebras. Then the family of dual maps

$$\Phi_{V'}^* : \mathcal{D}_Y(V')^\wedge \longrightarrow \mathcal{D}_X(\Phi^{0^{-1}}(V'))^\wedge$$

defines the morphism of sheaves of $\mathbb{Z}_2$-graded algebras $\Phi^* = (\Phi^0, \Phi^*) : \mathcal{D}_U^\wedge \longrightarrow \mathcal{D}_V^\wedge$. One has the following.

Proposition 3.5.4. *The correspondence*

$$\begin{aligned} \mathcal{D}_X(U) &\longrightarrow \mathcal{D}_U^\wedge = (U, \mathcal{D}_X(\cdot)^\wedge), \\ \Phi &\longrightarrow \Phi^* = (\Phi^0, \Phi^*), \end{aligned}$$

is a covariant functor from the model category **sc** *of open coalgebras to the category of sheavs of $\mathbb{Z}_2$-graded algebras.*

The functor above restricted to the subcategory $\mathbf{sc}^{<}$ *of finite-dimensional open coalgebras yields an inverse to the functor of Proposition* 3.4.4.

4. Smooth coalgebras

4.1. Category of smooth coalgebras

Definition 4.1.1. Let $\mathcal{M}$ be a pointed $\mathbb{Z}_2$-graded cocommutative coalgebra and $X = X_0 \oplus X_1$ a $\mathbb{Z}_2$-graded Fréchet space. An X-atlas on $\mathcal{M}$ is a collection $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ of charts (U_α, Φ_α) satisfying the following conditions:

1. The collection $\{U_\alpha\}_{\alpha \in I}$ is a covering of the set M of group-like elements of $\mathcal{M}$

$$M = \bigcup_{\alpha \in I} U_\alpha.$$

2. For each $\alpha \in I$, let $\mathcal{M}(U_\alpha)$ be a subcoalgebra of $\mathcal{M}$ given by

$$\mathcal{M}(U_\alpha) = \bigoplus_{p \in U_\alpha} \mathcal{M}_p,$$

   where $\mathcal{M}_p$ denotes the irreducible components of $\mathcal{M}$ containing $p \in U_\alpha$. Each Φ_α is an isomorphisms of $\mathbb{Z}_2$-graded coalgebras

$$\Phi_\alpha : \mathcal{M}(U_\alpha) \longrightarrow \mathcal{D}_X(\Phi^0(U_\alpha)),$$

   where $\mathcal{D}_X(\Phi^0(U_\alpha))$ is an open subcoalgebra of $\mathcal{D}_X$.
3. For any $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$, $\Phi^0_\alpha(U_\alpha \cap U_\beta)$ is an open subset of X_0, and

$$\Phi_\alpha \circ \Phi_\beta^{-1} : \mathcal{D}_X(\Phi^0_\beta(U_\alpha \cap U_\beta)) \longrightarrow \mathcal{D}_X(\Phi^0_\alpha(U_\alpha \cap U_\beta))$$

   is a diffeomorphism of open subcoalgebras of $\mathcal{D}_X$.

Let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ be an X-atlas of $\mathcal{M}$. For each $\alpha \in I$ the underlying part of Φ_α

$$\Phi^0_\alpha : U_\alpha \longrightarrow X_0$$

is a bijective map onto an open subset of X_0, and the compositions

$$\Phi^0_\alpha \circ (\Phi^0_\beta)^{-1} : \Phi^0_\beta(U_\alpha) \longrightarrow \Phi^0_\alpha(U_\alpha)$$

are homeomorphisms. As in the standard theory of manifolds [20] one easily shows that there exists a unique topology $\mathcal{T}_M$ on M such that all U_α are open and all Φ^0_α are homeomorphisms onto open subset of X_0.

Definition 4.1.2. Two X-atlases $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$, $\{(U_\beta, \Phi_\beta)\}_{\beta \in J}$ on $\mathcal{M}$ are compatible if the union of them $\{(U_\gamma, \Phi_\gamma)\}_{\gamma \in I \cup J}$ is an X-atlas on $\mathcal{M}$.

One easily verifies that compatible X-atlases on $\mathcal{M}$ determine the same topology $\mathcal{T}_M$ on M. Since the smoothness of a graded coalgebra morphism in **sc** is a local property one also has the following:

Proposition 4.1.1. *The relation of compatibility of X-atlases is an equivalence relation.*

Definition 4.1.3. Let $\mathcal{M}$ be a pointed $\mathbb{Z}_2$-graded cocommutative coalgebra and X a $\mathbb{Z}_2$-graded Fréchet space. An equivalence class of compatible X-atlases on $\mathcal{M}$ is called on X-smooth structure on $\mathcal{M}$. A coalgebra $\mathcal{M}$ with an X-smooth structure inducing a Hausdorff topology $\mathcal{T}_M$ on M is called a smooth coalgebra modelled on the $\mathbb{Z}_2$-graded Fréchet space X, or simply an X-smooth coalgebra.

Note that by Proposition 4.1.1 an X-atlas on $\mathcal{M}$ uniquely defines a smooth structure on $\mathcal{M}$.

Definition 4.1.4. Let $\mathcal{M}$ be an X-smooth coalgebra. An X-atlas on $\mathcal{M}$ is said to be admissible if it defines an original X-smooth structure on $\mathcal{M}$. A chart (U_α, Φ_α) on $\mathcal{M}$ is called admissible if it belongs to an admissible atlas on $\mathcal{M}$.

Remark 4.1.1. Let $\{(\mathcal{U}_\alpha, \Phi_\alpha)\}_{\alpha\in I}$ be an admissible X-atlas on a smooth coalgebra $\mathcal{M}$. By Proposition 3.2.2 and Theorem 3.2.2, $\{(U_\alpha, \Phi^0_\alpha)\}_{\alpha\in I}$ is a smooth X_0-atlas on M. By the same token compatible atlases on $\mathcal{M}$ induce compatible smooth atlases on M. The space M of group-like elements of $\mathcal{M}$ with the smooth structure determined by the atlas $\{(U_\alpha, \Phi^0_\alpha)\}_{\alpha\in I}$ is called *the underlying manifold of $\mathcal{M}$*.

Definition 4.1.5. Let $\mathcal{M}, \mathcal{N}$ be smooth coalgebras. A morphism $\Phi : \mathcal{M} \to \mathcal{N}$ of $\mathbb{Z}_2$-graded coalgebras is said to be smooth if for each $p \in M$ there exist admissible charts (U_α, Φ_α) on $\mathcal{M}$ and (V_γ, Ψ_γ) on $\mathcal{N}$ such that $p \in U_\alpha$, $\Phi^0(U_\alpha) \subset V_\gamma$, and the map

$$\Psi_\gamma \circ \Phi \circ \Phi_\alpha^{-1} : \mathcal{D}_X(U_\alpha) \longrightarrow \mathcal{D}_X(V_\gamma)$$

is a smooth morphism of open subcoalgebras.

As a simple consequence of Theorem 3.2.2 one gets:

Proposition 4.1.2. *The composition of smooth morphisms of smooth coalgebras is smooth.*

Definition 4.1.6. The objects of the category **SC** of smooth coalgebras are X-smooth coalgebras, where X runs over the category of graded Fréchet spaces.

For any two objects $\mathcal{M}, \mathcal{N} \in$ **OSC** the space of morphisms **MSC**$(\mathcal{M}, \mathcal{N})$ consists of all smooth morphisms of graded coalgebras.

The composition of morphisms in **SC** is defined as a composition of graded coalgebra morphisms.

An isomorphism in the category **SC** is called a diffeomorphism of smooth coalgebras.

Remark 4.1.2. In order to simplify further considerations we assume that objects and morphisms of the category **SC** are defined up to diffeomorphisms of smooth coalgebras with underlying parts being identitical maps in **FM**.

Remark 4.1.3. Let $\mathcal{M}, \mathcal{N}$ be smooth coalgebras, and $\Phi : \mathcal{M} \to \mathcal{N}$ a morphism of $\mathbb{Z}_2$-graded coalgebras. The restriction of Φ to the set M of group-like elements of $\mathcal{M}$, $\Phi^0 : M \to N$, is called *the underlying part* of the morphism Φ: For composition of two $\mathbb{Z}_2$-graded coalgebra morphisms one has $(\Psi \circ \Phi)^0 = \Psi^0 \circ \Phi^0$. By Remark 4.1.1 if (U_α, Φ_α), (V_γ, Ψ_γ) are admissible charts on $\mathcal{M}$ and $\mathcal{N}$, then $(U_\alpha, \Phi^0_\alpha)$, $(V_\gamma, \Psi^0_\gamma)$ are admissible charts on M and N, and the map

$$\Psi^0_\gamma \circ \Phi^0 \circ \Phi^{0-1}_\alpha : \Phi^0_\alpha(U_\alpha) \longrightarrow \Psi^0_\gamma(V_\gamma)$$

is smooth by Proposition 3.2.2. It follows that the underlying map of a smooth morphism of graded coalgebras is a smooth map of Fréchet manifolds.

Definition 4.1.7. Let $X = X_0 \oplus X_1$ be a $\mathbb{Z}_2$-graded Fréchet space, and $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ and admissible atlas on a smooth manifold M modelled on the Fréchet space X_0. Let $\{\Psi_{\alpha\beta}\}_{\alpha \in I}$ be a collection of maps such that

1. for all $\alpha \in I$, $\beta \in I(\alpha) \equiv \{\beta \in I : U_\alpha \cap U_\beta \neq \emptyset\}$

$$\Psi_{\alpha\beta} : \mathcal{D}_X(\varphi_\alpha(U_\alpha \cap U_\beta)) \longrightarrow \mathcal{D}_X(\varphi_\beta(U_\alpha \cap U_\beta))$$

   is an isomorphism of smooth open coalgebras such that $\Psi^0_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$;
2. for all $\alpha \in I$, $\Psi_{\alpha\alpha} = \mathrm{id}_{\mathbb{R}\varphi_\alpha(U_\alpha) \otimes S(X)}$;
3. for all $\alpha, \beta, \gamma \in I$ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$,

$$\Psi_{\beta\gamma} \circ \Psi_{\alpha\beta} = \Psi_{\alpha\gamma}$$

   on $\mathcal{D}_X(\varphi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma))$.

A collection $\{\Psi_{\alpha\beta}\}_{\alpha \in I}$ with the properties above is called an X-cocycle of transition **sc**-morphisms over the atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ on M.

One has the following "reconstruction theorem" which is very useful in constructing new smooth coalgebras,

Proposition 4.1.3. *Let $\{\Psi_{\alpha\beta}\}_{\alpha \in I}$ be an X-cocycle of transition* **sc**-*morphisms on M. Then there exists a unique smooth coalgebra $\mathcal{M}$ with the underlying manifold M and with the admissible X-atlas $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in I}$ such that*

$$\Psi_{\alpha\beta} = \Psi_\beta \circ \Psi_\alpha^{-1}{}_{|\mathcal{D}_X(\Psi^0_\alpha(U_\alpha \cap U_\beta))} \tag{35}$$

for all $\alpha \in I$, $\beta \in I(\alpha)$.

The smooth coalgebra $\mathcal{M}$ and the X-atlas $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in I}$ of the proposition above are said to be *generated* by the X-cocycle $\{\Psi_{\alpha\beta}\}_{\alpha \in I}$.

Proof. Let $\{\Psi_{\alpha\beta}\}_{\alpha\in I}$ be an X-cocycle related to a smooth atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ on M. For $p \in M$, consider the space of pairs $(\alpha, m)_p$, where $\alpha \in I$ is such that $p \in U_\alpha$ and $m \in \mathcal{D}_{X\varphi_\alpha(p)} = \mathbb{R}\{\varphi_\alpha(p)\} \otimes S(X)$. Two such pairs are equivalent

$$(\alpha, m)_p \sim (\alpha', m')_p$$

if $m' = \Psi_{\alpha\alpha'}(m)$. Using the cocycle properties (Definition 4.1.7) one easily shows this is an equivalence relation. Let $\mathcal{M}_p$ denote the space of equivalence classes of this relation and let $[(\alpha, m)_p]_\sim$ denote the equivalence class of $(\alpha, m)_p$. Since $\Psi_{\alpha\alpha'}$ are $\mathbb{Z}_2$-graded coalgebra morphisms $\mathcal{M}_p$ acquires the structure of irreducible pointed $\mathbb{Z}_2$-graded cocomutative coalgebra:

$$\Delta[(\alpha, m)_p]_\sim \equiv \sum_{(m)}[(\alpha, m_{(1)})_p]_\sim \otimes [(\alpha, m_{(2)})_p]_\sim,$$

$$\varepsilon([(\alpha, m)_p]_\sim) \equiv \varepsilon_{\varphi_\alpha(U_\alpha)}(m).$$

Let $\mathcal{M}$ be the direct sum of irreducible $\mathbb{Z}_2$-graded coalgebras

$$\mathcal{M} \equiv \bigoplus_{p\in M} \mathcal{M}_p.$$

For each $\alpha \in I$ we define

$$\Psi_\alpha : \mathcal{M}(U_\alpha) \ni [(\alpha, m)_u]_\sim \longrightarrow m \in \mathbb{R}\varphi_\alpha(U_\alpha) \otimes S(X).$$

Using the cocycle properties (Definition 4.1.7) one verifies that the collection $\{(U_\alpha, \Psi_\alpha)\}_{\alpha\in I}$ is an X-atlas on $\mathcal{M}$ with the required transition **sc**-morphisms. By Proposition 4.1.1 it defines a unique X-smooth structure on $\mathcal{M}$.

Suppose that there exists another smooth coalgebra $\mathcal{M}'$ over M with an admissible X-atlas $\{(U'_\alpha, \Psi'_\alpha)\}_{\alpha\in I}$ satisfying condition (35). Then the map defined for each $p \in M$ by

$$\mathcal{M}'_p \ni m \longrightarrow \Psi_\alpha^{-1} \circ \Psi'_\alpha \in \mathcal{M}_p$$

extends by linearity to the diffeomorphism of smooth coalgebras over the identity map. Hence $\mathcal{M}' = \mathcal{M}$ by Remark 4.1.2. □

Two cocycles $\{\Psi'_{\alpha'\beta'}\}_{\alpha'\in I}$, $\{\Psi''_{\alpha''\beta''}\}_{\alpha''\in I''}$ of transition **sc**-morphisms on M are said to be *compatible* if there exists a third one $\{\Psi_{\alpha\beta}\}_{\alpha\in I}$ such that

$$\{\varphi'_{\alpha'}\}_{\alpha'\in I'} \cup \{\varphi''{}_{\alpha''}\}_{\alpha''\in I''} \subset \{\varphi_\alpha\}_{\alpha\in I},$$

$$\{\Psi'_{\alpha'\beta'}\}_{\alpha'\in I'} \cup \{\Psi''{}_{\alpha''\beta''}\}_{\alpha''\in I''} \subset \{\Psi_{\alpha\beta}\}_{\alpha\in I},$$

as sets of maps. Using the construction of Proposition 4.1.3 for all three cocycles of the definition above and comparing the resulting smooth coalgebras one gets the following result.

Proposition 4.1.4. *Compatible X-cocycles of transition* **sc**-*morphisms on M generate the same X-smooth coalgebra $\mathcal{M}$ and compatible X-atlases on $\mathcal{M}$.*

Definition 4.1.8. Let $\mathcal{M}$ be an X-smooth coalgebra. A linear functional f on $\mathcal{M}$ is called a superfunction if for each $p \in M$ there exists an admissible chart (U_α, Φ_α) such that $p \in U_\alpha$, and the functional $(\Phi_\alpha^{-1})' f$ is a superfunction on the open coalgebra $\mathcal{D}_X(\Phi_\alpha^0(U_\alpha))$.

Proposition 4.1.5. *Let $\mathcal{M}'$ be the full algebraic dual of $\mathcal{M}$ endowed with the $\mathbb{Z}_2$-graded algebra structure dual to the $\mathbb{Z}_2$-graded coalgebra structure on $\mathcal{M}$.*

The subspace $\mathcal{M}^\wedge \subset \mathcal{M}'$ consisting of all superfunctions on an X-smooth coalgebra $\mathcal{M}$ is a $\mathbb{Z}_2$-graded subalgebra of $\mathcal{M}'$

Let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in I}$ be an admissible X-atlas on a smooth coalgebra $\mathcal{M}$ and let U be an open (with respect to the induced topology $\mathcal{T}_M$) subset of M. Then the collection

$$\{(U \cap U_\alpha, \Phi_{\alpha|\mathcal{M}(U)\cap\mathcal{M}(U_\alpha)})\}_{\alpha\in I} \tag{36}$$

is an X-atlas on the subcoalgebra

$$\mathcal{M}(U) = \bigoplus_{p\in U} \mathcal{M}_p.$$

Compatible X-atlases on $\mathcal{M}$ induce compatible X-atlases on $\mathcal{M}(U)$. The subcoalgebra $\mathcal{M}(U) \subset \mathcal{M}$ with the smooth structure defined by the induced atlas (36) is called an *open subcoalgebra* of $\mathcal{M}$.

Assigning to each open subset $U \subset M$ the $\mathbb{Z}_2$-graded algebra $\mathcal{M}(U)^\wedge$ of superfunctions on $\mathcal{M}(U)$ and introducing the restriction maps as duals to the inclusions $\mathcal{M}(U') \subset \mathcal{M}(U)$, $(U' \subset U)$ one gets a sheaf of $\mathbb{Z}_2$-graded algebras $\mathcal{M}_M^\wedge = (M, \mathcal{M}(.)^\wedge)$. $\mathcal{M}_M^\wedge$ is called the *sheaf of superfunctions on* $\mathcal{M}$.

Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of smooth coalgebras. By Remark 3.5.2 for each superfunction $g \in \mathcal{N}^\wedge$, the functional $\Phi' g$ is a superfunction on $\mathcal{M}$. $\Phi^* g = \Phi' g$ is called the *pull-back* of g. For each open subset $V \subset N$ we define $\Phi_V^* : \mathcal{N}(V)^\wedge \to \mathcal{M}(U)^\wedge$ as a map dual to

$$\Phi_V : \mathcal{M}(\Phi^{0-1}(V) \ni \mu \longrightarrow \Phi(\mu) \in \mathcal{N}(V).$$

The collection of maps $\{\Phi^*\}$ defines a morphism of sheaves of $\mathbb{Z}_2$-graded algebras $\Phi^* = (\Phi^0, \Phi^*.) : \mathcal{M}_M^\wedge \longrightarrow \mathcal{N}_N^\wedge$. One has the following global version of Proposition 3.5.4.

Proposition 4.1.6. *The correspondence*

$$\mathcal{M} \longrightarrow \mathcal{M}_M^\wedge = (M, \mathcal{M}(.)^\wedge),$$
$$\Phi \longrightarrow \Phi^+ = (\Phi^0, \Phi^*_.)$$

is a covariant functor from the category **SC** *of smooth coalgebras to the category of sheaves of $\mathbb{Z}_2$-graded algebras.*

4.2. Direct product

Let $\mathcal{M}, \mathcal{N}$ be smooth coalgebras modelled on graded Fréchet spaces X, and Y, respectively. One can introduce an $(X \oplus Y)$-smooth structure on $\mathcal{M} \otimes \mathcal{N}$ as follows. Let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ be an admissible X-atlas on $\mathcal{M}$, and $\{(V_\beta, \Psi_\beta)\}_{\beta \in J}$ an admissible Y-atlas on $\mathcal{N}$. Consider the collection

$$\{(U_\alpha \times V_\beta, \Phi_\alpha \otimes \Psi_\beta)\}_{(\alpha,\beta) \in I \times J}. \tag{37}$$

One obviously has

$$M \times N = \bigcup_{(\alpha,\beta) \in I \times J} U_\alpha \times V_\beta,$$

and the $\mathbb{Z}_2$-graded coalgebra isomorphisms

$$\begin{aligned} &\mathcal{D}_{X \oplus Y}(\Phi^0_\alpha \times \Psi^0_\beta(U_\alpha \times V_\beta)) \\ &\quad = \mathcal{D}_X(\Phi^0_\alpha(U_\alpha)) \otimes \mathcal{D}_Y(\Psi^0_\beta(V_\beta)), \\ &\mathcal{D}_{X \oplus Y}(\Phi^0_\alpha \times \Psi^0_\beta((U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}))) \\ &\quad = \mathcal{D}_X(\Phi^0_\alpha(U_\alpha \cap U_{\alpha'})) \otimes \mathcal{D}_Y(\Psi^0_\beta(V_\beta \cap V_{\beta'})). \end{aligned}$$

By Definition 3.3.2 the RHS of the equations above can be regarded as tensor products in the model category while the LHS as open subcoalgebras of $\mathcal{D}_{X \oplus Y}$. Then the $\mathbb{Z}_2$-graded coalgebra morphisms

$$(\Phi_\alpha \otimes \Psi_\beta) \circ (\Phi_{\alpha'} \otimes \Psi_{\beta'})^{-1} = (\Phi_\alpha \circ \Phi^{-1}_{\alpha'}) \otimes (\Phi_\beta \circ \Phi^{-1}_{\beta'})$$

are smooth by Proposition 3.3.1. It follows that collection (37) defines an $(X \oplus Y)$-atlas on $\mathcal{M} \otimes \mathcal{N}$. One can easily verify that compatible atlases on $\mathcal{M}$ and $\mathcal{N}$ lead by the construction above to compatible atlases on $\mathcal{M} \otimes \mathcal{N}$, hence the following definition:

Definition 4.2.1. The tensor product of two smooth coalgebras $\mathcal{M}, \mathcal{N} \in$ **OSC** modelled on the $\mathbb{Z}_2$-graded Fréchet spaces X, Y, respectively, is the tensor product of $\mathbb{Z}_2$-graded cocommutative coalgebras $\mathcal{M} \otimes \mathcal{N}$ endowed with the $(X \oplus Y)$-smooth structure determined by the atlas (37).

As a consequence of Proposition 3.3.1 one gets:

Proposition 4.2.1.
1. *Let $\Phi : \mathcal{M} \to \mathcal{N}$, $\Phi' : \mathcal{M}' \to \mathcal{N}'$ be morphisms of smooth coalgebras. Then the tensor product of graded coalgebra morphisms*

$$\Phi \otimes \Phi' : \mathcal{M} \otimes \mathcal{M}' \ni m \otimes m' \longrightarrow \Phi(m) \otimes \Phi'(m') \in \mathcal{N} \otimes \mathcal{N}'$$

is a morphism of smooth coalgebras.

2. *Let $\mathcal{M} \otimes \mathcal{N}$ be the tensor product of smooth coalgebras. Let $\varepsilon_M, \varepsilon_N$ be counits in the coalgebras $\mathcal{M}, \mathcal{N}$, respectively. Then the maps*

$$P_M : \mathcal{M} \otimes \mathcal{N} \ni m \otimes n \longrightarrow \varepsilon_N(n) \cdot m \in \mathcal{M},$$
$$P_N : \mathcal{M} \otimes \mathcal{N} \ni m \otimes n \longrightarrow \varepsilon_M(m) \cdot n \in \mathcal{N}$$

   are morphisms of smooth coalgebras.
3. *The comultiplication $\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ in a smooth coalgebra $\mathcal{M}$ is a morphism of smooth coalgebras.*

Remark 4.2.1. Let us observe that the $X \oplus Y$-atlas (37) induces the smooth atlas

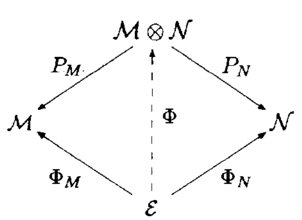
$$\{(U_\alpha \times V_\beta, \Phi^0_\alpha \times \Psi^0_\beta)\}_{(\alpha,\beta)\in I\times J}$$

on $M \times N$. It follows that the underlying manifold of $\mathcal{M} \otimes \mathcal{N}$ is the cartesian product $M \times N$ of Fréchet manifolds. The underlying parts of smooth coalgebra morphisms from Proposition 4.2.1 are given by

$$(\Phi \otimes \Phi')^0 : M \times M' \ni (u, u') \longrightarrow (\Phi^0(u), \Phi'^0(u')) \in N \times N',$$
$$P^0_M : M \times N \ni (u, v) \longrightarrow u \in M,$$
$$P^0_N : M \times N \ni (u, v) \longrightarrow v \in N,$$
$$\Delta^0 : M \ni u \longrightarrow (u, u) \in M \times M.$$

By Theorem 3.3.1 and Proposition 4.2.1 one gets

Theorem 4.2.1. *$(\mathcal{M} \otimes \mathcal{N}, P_M, P_N)$ is the direct product in the category of smooth coalgebras, i.e. for every smooth coalgebra $\mathcal{E}$ and smooth coalgebra morphisms $\Phi_M : \mathcal{E} \to \mathcal{M}$, $\Phi_N : \mathcal{E} \to \mathcal{N}$ there exists a uniqe morphism of smooth coalgebras $\Phi : \mathcal{E} \to \mathcal{M} \otimes \mathcal{N}$ making the diagram*



commute.

Remark 4.2.2. The unique morphism Φ in the theorem above is given by the composition

$$\Phi = (\Phi_M \otimes \Phi_N) \circ \Delta_E,$$

and its underlying part by

$$\Phi^0 : E \in u \longrightarrow (\Phi^0_M(u), \Phi^0_N(u)) \in M \times N.$$

4.3. Subcategory of smooth Fréchet manifolds

Let us denote by **OFM** and **MFM** the collections of objects and morphisms of the category **FM** of smooth Fréchet manifolds. In this Section we shall show that the category **FM** is equivalent to the category $\mathbf{SC}_0$ of even smooth coalgebras. $\mathbf{SC}_0$ is defined as the full subcategory of SC consisting of all smooth coalgebras modelled on purely even graded Fréchet spaces $X = X \oplus \{o\}$. By definition for all $\mathcal{M}, \mathcal{N} \in \mathbf{SC}_0$

$$\mathbf{MSC}_0(\mathcal{M}, \mathcal{N}) \equiv \mathbf{MSC}(\mathcal{M}, \mathcal{N}).$$

Note that by Theorem 4.2.1 $\mathbf{SC}_0$ inherits the direct product from **SC**.

Gathering together Remarks 4.1.1, 4.1.3, 4.2.1, and 4.2.2 one gets:

Proposition 4.3.1. *The correspondence*

$$\mathbf{OSC} \ni \mathcal{M} \longrightarrow M \in \mathbf{OFM},$$
$$\mathbf{MSC}(\mathcal{M}, \mathcal{N}) \in \Phi \longrightarrow \Phi^0 \in \mathbf{MFM}(M, N)$$

is a covariant functor respecting the direct product.

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be an admissible atlas on a Fréchet manifold M modelled on a Fréchet space X. By Remark 3.4.1 the family $\{\Phi_{\alpha\beta}\}_{\alpha\in I}$ of **sc**-morphisms defined by

$$\Psi_{\alpha\beta} = (\varphi_\beta \circ \varphi_\alpha^{-1})_{*|\mathbb{R}\Psi_\alpha^0(U_\alpha\cap U_\beta)\otimes S(X)}, \quad \beta \in I(\alpha) \tag{38}$$

is an $(X\oplus\{o\})$-cocycle of transition **sc**-morphisms over the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ on M. By the same token cocycles constructed from compatible smooth atlases on M by formula (38), are compatible. Thus, by Propositions 4.1.3 and 4.1.4, the following definition is not ambiguous.

Definition 4.3.1. The $(X \oplus \{o\})$-smooth coalgebra $\mathcal{D}(M)$ generated by the cocycle (38) is called the smooth coalgebra of the Fréchet manifold M.

For each admissible atlas $\{(U_{\alpha,\varphi\alpha})\}_{\alpha\in I}$ on M, the $(X \oplus \{0\})$-atlas on $\mathcal{D}(M)$ generated by the cocycle (38) will be denoted by $\{(U_\alpha, \varphi_{\alpha*})\}_{\alpha\in I}$. Applying this construction to the maximal atlas on M we define for each admissible chart (U, φ) on M the corresponding admissible chart (U, φ_*) on $\mathcal{D}(M)$. One easily verifies that the definition of (U, φ_*) is independent of the choice of admissible atlas containing (U, φ).

Proposition 4.3.2. *Let $\mathcal{M} \in \mathbf{OSC}_0$. Then $\mathcal{M}$ is the smooth coalgebra of its underlying manifold M, i.e. $\mathcal{M} = \mathcal{D}(M)$.*

Proof. Let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in I}$ be an admissible $(X \oplus \{o\})$-atlas on $\mathcal{M}$. Then by Remark 4.1.1 the atlas $\{(U_\alpha, \Phi_\alpha^0)\}_{\alpha\in I}$ is an admissible smooth atlas on the underlying manifold M. For all $\alpha, \beta \in I(\alpha)$, $\Phi_\alpha \circ \Phi_\beta^{-1}$ are morphisms of the model category $\mathbf{sc}_0$ and by Remark 3.4.1 $\Phi_\alpha \circ \Phi_\beta^{-1} = (\Phi_\alpha^0 \circ \Phi_\beta^{-1})_*$. Then by the construction of $\mathcal{D}(M)$ the map defined for each $p \in M$ by

$$\mathcal{M}_p \ni \mu \longrightarrow (\Phi_{\alpha*}^0)^{-1} \circ \Phi_\alpha(\mu) \in \mathcal{D}(M)_p$$

extends by linearity to a diffeomorphism of smooth coalgebras over the identity map. Here $\mathcal{M} = \mathcal{D}(M)$ by Remark 4.1.2. □

Let $\phi : M \longrightarrow N$ be a smooth map of Fréchet manifolds. Then for each $p \in M$ there are admissible charts $(U_\alpha, \varphi_\alpha)$ at $p \in M$ and (V_β, ψ_β) at $\psi(p) \in N$ such that the composition

$$\psi_\beta \circ \phi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \longrightarrow \psi_\beta(V_\beta)$$

is a smooth map between open subset of Fréchet spaces (morphism in **fm**).

For all $\mu \in \mathcal{D}(M)_p$ we define

$$\phi_{*p}(\mu) \equiv \psi_{\beta *}^{-1} \circ (\psi_\beta \circ \phi \circ \varphi_\alpha^{-1})_* \circ \varphi_{\alpha *}(\mu). \tag{39}$$

By Remark 3.4.1 the definition above is independent of the choice of admissible charts at $p \in M$ and $\phi(p) \in N$. Extending formula (39) by linearity in p one gets the smooth morphism of graded coalgebras

$$\phi_* : \mathcal{D}(M) \longrightarrow \mathcal{D}(N),$$

with underlying part $(\phi_*)^0 = \phi$.

Proposition 4.3.3. *Let* $\mathcal{M}, \mathcal{N} \in \mathbf{OSC}_0$. *For all smooth morphisms of graded coalgebras* $\Phi : \mathcal{M} \to \mathcal{N}$, $\Phi = (\Phi^0)_*$.

Proof. By Definition 4.1.5 for each $p \in M$ there exist admissible charts (U_α, Φ_α) on $\mathcal{M}$, and (V_γ, Ψ_γ) on $\mathcal{N}$, such that $p \in U_\alpha$, $\Phi^0(U_\alpha) \subseteq V_\gamma$ and the composition $\Psi_\gamma \circ \Phi \circ (\Phi_\alpha)^{-1}$ is an $\mathbf{sc}_0$-morphism. Then by Remark 3.4.1

$$\Psi_\gamma \circ \Phi \circ (\Phi_\alpha)^{-1} = (\Psi_\gamma^0 \circ \Phi^0 \circ (\Phi_\alpha^0)^{-1})_*.$$

By Proposition 4.3.2 one can assume $\Phi_\alpha = \Phi_{\alpha *}^0$ and $\Psi_\gamma = \Psi_{\gamma *}^0$. Hence for all $p \in M$, $\mu \in \mathcal{D}(M)_p$

$$\Phi(\mu) = (\Psi_{\gamma *}^0)^{-1} \circ (\Psi_\gamma^0 \circ \Phi^0 \circ (\Phi_\alpha^0)^{-1})_* \circ \Phi_{\alpha *}^0(\mu),$$

and $\Phi = (\Phi^0)_*$. □

Propositions 4.3.1–4.3.3 imply the following global counterpart of Proposition 3.4.1 and Remark 3.4.1.

Theorem 4.3.1. *The correspondence*

$$\mathbf{OFM} \ni M \longrightarrow \mathcal{D}(M) \in \mathbf{OSC}_0,$$
$$\mathbf{OFM}(M, N) \ni \phi \longrightarrow \phi_* \in \mathbf{MSC}_0(\mathcal{D}(M), \mathcal{D}(N)),$$

is an equivalence of categories **FM** *and* $\mathbf{SC}_0$. *Moreover it is the right inverse to the functor of Proposition* 4.3.1.

The result above means that the category of smooth Fréchet manifolds can be regarded as a full subcategory of the category of smooth coalgebras. This partly justifies our construction of **SC** as a $\mathbb{Z}_2$-graded extension of **FM**.

Remark 4.3.1. Composing the functors from Proposition 4.3.1 and Theorem 4.3.1 one gets the covariant functor respecting the direct product

$$\mathbf{OSC} \ni \mathcal{M} \longrightarrow M \longrightarrow \widetilde{\mathcal{M}} \equiv \mathcal{D}(M) \in \mathbf{OSC}_0,$$
$$\mathbf{MSC}(\mathcal{M}, \mathcal{N}) \ni \Phi \longrightarrow \Phi^0 \longrightarrow \widetilde{\Phi} \equiv (\Phi^0)_* \in \mathbf{MSC}_0(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}).$$

The functor above is called *the underlying functor.*

Let us note that $\widetilde{\mathcal{M}}$ is canonically embedded in $\mathcal{M}$. Let $i_0 : S(X_0) \longrightarrow S(X_0 \otimes X_1)$ be the canonical embedding defined as the universal extension of the composition

$$X_0 \xrightarrow{\rho_0} X_0 \oplus X_1 \xrightarrow{0} S(X_0 \oplus X_1).$$

For each $p \in M$ we define

$$\widetilde{\mathcal{M}}_p \ni \mu \longrightarrow \Phi_\alpha^{-1} \circ (\mathrm{id}_{\Phi_\alpha^0(U_\alpha)} \otimes i_0) \circ \Phi_{\alpha *}^0(\mu) \in \mathcal{M}_p,$$

where (U_α, Φ_α) is an admissible chart of $\mathcal{M}$ at $p \in M$. One easily verifies that the definition above is independent of the choice of an admissible chart at p and extends by linearity to the morphism of smooth coalgebras $\tilde{i} : \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$, with $\tilde{i}^0 = \mathrm{id}_M$. By Remark 4.1.2 $\widetilde{\mathcal{M}}$ can be regarded as a subcoalgebra of $\mathcal{M}$. $\widetilde{\mathcal{M}}$ is called the *underlying subcoalgebra of* $\mathcal{M}$.

Remark 4.3.2. We shall briefly discuss the geometric interpretation of the smooth coalgebra of a Fréchet manifold M. By definition, $\mathcal{D}(M)$ is the direct sum of its irreducible components

$$\mathcal{D}(M) = \bigoplus_{p \in M} \mathcal{D}(M)_p.$$

With respect to the graded coalgebra structure each irreducible component $\mathcal{D}(M)_p$ is isomorphic with $S(X \oplus \{0\})$. For each $p \in M$ let

$$\mathcal{D}(M)_p = \bigcup_{k \geq 0} \mathcal{D}(M)_p^{(k)}$$

be the coardical filtration of $\mathcal{D}(M)_p$.

The smooth structure on $\mathcal{D}(M)$ is related to the smooth structures of the kth order co-jet vector bundles over M in the following way. For $k \geq 0$ we define the subset

$$\mathcal{T}^{(k)}(M) \equiv \bigcup_{p \in M} \mathcal{D}(M)_p^{(k)} x \subset \mathcal{D}(M),$$

and the projection

$$\pi^{(k)} : \mathcal{T}^{(k)}(M)_p^{(k)} \supset \mathcal{D}(M)_p^{(k)} \ni \mu \longrightarrow p \in M.$$

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be an admissible atlas on M and $\{(U_\alpha, \varphi_{\alpha*})\}_{\alpha\in I}$ the corresponding $(X \oplus \{o\})$-atlas on $\mathcal{D}(M)$. For each $k \geq 0$, the collection $\{(U_\alpha, \tau_\alpha^{(k)})\}_{\alpha\in I}$, where

$$\tau_\alpha^{(k)} : (\pi^{(k)})^{-1}(U_\alpha) \ni \mu \longrightarrow \varphi_{\alpha*}(\mu) \in U_\alpha \times S^{(k)}(X), \tag{40}$$

is a trivialising covering of $\pi^{(k)} : \tau^{(k)}(M) \to M$ (we are using the terminology of [20]). One easily verifies that compatible atlases on $\mathcal{D}(M)$ yield compatible trivialising coverings and that $\pi^{(k)} : \tau^{(k)}(M) \to M$ acquires the structure of a smooth vector bundle over M with standard fibre $S^{(k)}(X)$.

In the case of a finite-dimensional Fréchet space $X \approx \mathbb{R}^n$ the bundle $\pi^{(k)} : \tau^{(k)}(M) \to M$ constructed above is called the kth order co-jet bundle over M and is dual to the bundle of kth order jets on M.

In the case of an infinite-dimensional Fréchet space X, for $k \geq 2$ the standard fibre $S^{(k)}(X)$ is not complete with respect to the direct sum and the projective tensor product topologies on

$$S^{(k)}(X) = \bigoplus_{i=0}^{k} S^i(X),$$

and $S^i(X)$, respectively. Since the transition maps of the trivialising covering (40) are continuous with respect to this topology, the bundle $\pi^{(k)} : \tau^{(k)}(M) \to M$ admits a unique extension to a bundle with a complete standard fibre. Let us stress that in the present coalgebraic approach this completion will not be used.

Remark 4.3.3. The construction discussed in the previous remark applies also to the subset of all primitive elements of the coalgebras $\mathcal{D}(M)$. Let $\tau_p(M)$ be the space of all primitive, with respect to $p \in M$, elements of $\mathcal{D}(M)$. For each $p \in M$ one has the invariant decomposition

$$\mathcal{D}(M)_p^{(1)} = \mathbb{R} \oplus \mathcal{T}_p(M).$$

Let us introduce the set

$$\mathcal{T}(M) \equiv \bigcup_{p\in M} \mathcal{T}_p(M),$$

with the projection $\pi : \mathcal{T}(M) \to M$ given by $\pi(\mathcal{T}_p(M)) = p$. In this case the smooth structure on $\mathcal{D}(M)$ induces the canonical smooth structure of the tangent bundle of M.

Let M be a smooth Fréchet manifold, $(\mathcal{D}(M), \Delta_M, \varepsilon_M)$ the smooth coalgebra of M, and $(C^\infty(M), M, u)$ the algebra of smooth functions on M. For each $p \in M$ let us consider the pairing

$$\langle\cdot,\cdot\rangle_p : \mathcal{D}(M)_p \times C^\infty(M) \longrightarrow \mathbb{R},$$

given by

$$\langle\mu_p, f\rangle_p \equiv \langle\varphi_{\alpha*}(\mu_p), f \circ \varphi_\alpha^{-1}\rangle_{\varphi_\alpha(U_\alpha)}, \tag{41}$$

where $(U_\alpha, \varphi_\alpha)$ is an admissible chart on M at p. One easily verifies that the definition above is independent of the choice of an admissible chart at p. Extending formula (41) by linearity in the first variable one gets the pairing

$$\langle\cdot,\cdot\rangle_M : \mathcal{D}(M) \times C^\infty(M) \longrightarrow \mathbb{R}. \tag{42}$$

Using Remark 3.4.1 and Proposition 3.4.2 one gets the following.

Proposition 4.3.4. *Let $\mathcal{D}(M)$ be the smooth coalgebra of a Fréchet manifold M and $C^\infty(M)$ the algebra of smooth functions on M.*

1. *For all $f, g \in C^\infty(M)$, $\mu \in \mathcal{D}(M)$*

$$\langle\mu, f \cdot g\rangle_M = \sum_{(\mu)} \langle\mu_{(1)}, f\rangle_M \langle\mu_{(2)}, g\rangle_M,$$
$$\langle\mu, 1\rangle_M = \varepsilon_M(\mu),$$

where $\Delta_M \mu = \sum_{(\mu)} \mu_{(1)} \times \mu_{(2)}$.

2. *Let $\phi : M \to N$ be a smooth map of Fréchet manifolds. Then*

$$\langle\phi^* \mu, f\rangle_N = \langle\mu, f \circ \phi\rangle_M,$$

for all $\mu \in \mathcal{D}(M)$, $f \in C^\infty(N)$.

As in the case open even coalgebras one can show that the pairing (42) is nonsingular. Then the proposition above implies the following.

Theorem 4.3.2. *The map*

$$\mathcal{D}(M) \ni \mu \longrightarrow \langle\mu, \cdot\rangle \in C^\infty(M)^\circ$$

is an injective morphism of $\mathbb{Z}_2$-graded coalgebras.

4.4. Subcategory of supermanifolds

We define the category $\mathbf{SC}^<$ as a subcategory of $\mathbf{SC}$ consisting of all smooth coalgebras modelled on finite-dimensional $\mathbb{Z}_2$-graded Fréchet spaces and all $\mathbf{SC}$-morphisms between them. By definition $\mathbf{SC}^<$ is a full subcategory of $\mathbf{SC}$, i.e. for all $\mathcal{M}, \mathcal{N} \in \mathbf{OSC}^<$

$$\mathbf{MSC}^<(\mathcal{M}, \mathcal{N}) = \mathbf{MSC}(\mathcal{M}, \mathcal{N}).$$

By Theorem 4.2.1 $\mathbf{SC}^<$ inherits the direct product from $\mathbf{SC}$. In this section we shall construct an equivalence of the category of BLK supermanifolds $\mathbf{SM}$ introduced in Section 2.4 with the category of finite-dimensional smooth coalgebras $\mathbf{SC}^<$.

Let $\mathcal{A}_M$ be a supermanifold. For each (m,n)-atlas $\{(U_\alpha, F_\alpha)\}_{\alpha\in I}$ on $\mathcal{A}_M$ the family of maps $\{F_{\alpha\beta}\}_{\alpha\in I}$ given for all $\alpha \in I$, $\beta, \in I(\alpha)$ by

$$F_{\alpha\beta} = F_\beta \circ F_\alpha^{-1} : \mathcal{S}^{m,n}_{F_\alpha^0(U_\alpha\cap U_\beta)} \longrightarrow \mathcal{S}^{m,n}_{F_\beta^0(U_\alpha\cap U_\beta)} \tag{43}$$

is an (m,n)-cocycle of **sm**-transition maps over the smooth atlas $\{(U_\alpha, F^0_\alpha)\}_{\alpha\in I}$ on the underlying manifold M. Applying the functor of Proposition 3.4.4 to the cocycle (43) one gets $(\mathbb{R}^m \oplus \mathbb{R}^n)$-cocycle of **sc**-transition maps

$$F_{\alpha\beta *} = (F_\beta \circ F_\alpha^{-1})_* : \mathcal{D}_{m,n}(F^0_\alpha(U_\alpha \cap U_\beta)) \longrightarrow \mathcal{D}_{m,n}(F^0_\beta(U_\alpha \cap U_\beta)), \tag{44}$$

over the same smooth atlas on M. By Proposition 4.1.3 the cocycle (44) generates a unique smooth coalgebra $\mathcal{D}(\mathcal{A}_M)$ and the admissible $(\mathbb{R}^m \oplus \mathbb{R}^n)$-atlas $\{(U_\alpha, F_{\alpha *})\}_{\alpha\in I}$ on $\mathcal{D}(\mathcal{A}_M)$. One easily verifies that different (m,n)-atlases on $\mathcal{A}_M$ lead by the above construction to compatible $(\mathbb{R}^m \oplus \mathbb{R}^n)$-atlases on the same smooth coalgebra $\mathcal{D}(\mathcal{A}_M)$. Applying the construction to the maximal (m,n)-atlas on $\mathcal{A}_M$, we define for each chart (U, F) on $\mathcal{A}_M$ the corresponding admissible chart (U, F_*) on $\mathcal{D}(\mathcal{A}_M)$.

Definition 4.4.1. The smooth coalgebra $\mathcal{D}(\mathcal{A}_M)$ generated by the cocycle (44) is called the smooth coalgebra of the supermanifold $\mathcal{A}_M$.

Reversing the construction above and using the reconstruction theorem for supermanifolds (Proposition 2.4.3) one can show that the correspondence

$$\mathbf{OSM} \ni \mathcal{A}_M \longrightarrow \mathcal{D}(\mathcal{A}_M) \in \mathbf{OSC}^<$$

is bijective. Moreover for each admissible chart (U, Φ) on $\mathcal{D}(\mathcal{A}_M)$ there exists a chart (U, F) on $\mathcal{A}_M$ such that $\Phi = F_*$.

Let $F = (F^0, F_\cdot) : \mathcal{A}_M \to \mathcal{B}_N$ be a morphism of supermanifolds. Then for each $p \in M$ there are charts (U_α, F_α) on $\mathcal{A}_M$ and (V_γ, G_γ) on $\mathcal{B}_N$ such that $p \in U_\alpha$, $F^0(U_\alpha) \subset V_\gamma$, and the composition

$$G_\gamma \circ F \circ F_\alpha^{-1} : \mathcal{S}_{m,n}(F^0_\alpha(U_\alpha)) \longrightarrow \mathcal{S}_{m,n}(G^0_\gamma(V_\gamma))$$

is a morphism in the model category **sm**.

Let $\mathcal{D}(\mathcal{A}_M)_p$ be the irreducible component of $\mathcal{D}(\mathcal{A}_M)$ containing the group-like element $p \in M$. For all $\mu \in \mathcal{D}(\mathcal{A}_M)_p$ we define

$$F_{*p}(\mu) \equiv G^{-1}_{\gamma *} \circ (G_\gamma \circ F \circ F_\alpha^{-1})_* \circ F_{\alpha *}(\mu). \tag{45}$$

By Proposition 3.4.4 the definition above is independent of the choice of admissible charts at $p \in M$ and $\Phi^0(p) \in N$. Extending formula (45) by linearity in p one gets the smooth morphism of graded coalgebras

$$F_* : \mathcal{D}(\mathcal{A}_M) \longrightarrow \mathcal{D}(\mathcal{B}_N),$$

with the underlying part $(F_*)^0 = F^0$. As a consequence of Proposition 3.4.4 one gets:

Theorem 4.4.1. *The correspondence*

$$\begin{aligned} \mathbf{OSM} \ni \mathcal{A}_M &\longrightarrow \mathcal{D}(\mathcal{A}_M) \in \mathbf{OSC}^<, \\ \mathbf{OSM}(\mathcal{A}_M, \mathcal{B}_N) \ni F &\longrightarrow F_* \in \mathbf{MSC}^<(\mathcal{D}(\mathcal{A}_M), \mathcal{D}(\mathcal{B}_N)) \end{aligned} \tag{46}$$

is an equivalence of categories **SM** *and* $\mathbf{SC}^<$.

It follows that the category of BLK supermanifolds can be identified with the full subcategory of the category of finite-dimensional smooth coalgebras. Since by Theorem 4.3.1 the category of Fréchet manifolds can be identified with subcategory of even smooth coalgebras, **SC** provides a correct extension of both categories.

Remark 4.4.1. As in the case of category **FM** of Fréchet manifolds (Remarks 4.3.2 and 4.3.3) the smooth coalgebra of a supermanifold admits the geometrical interpretation in terms of co-jet vector superbundles [5]. Since the constructions of these bundles requires some techniques of algebraic geometry [17] not used in the present paper, we refer to the original paper [5] for the discussion of this point.

Let $(\mathcal{D}(\mathcal{A}_M), \Delta_M, \varepsilon_M)$ be the smooth coalgebra of a supermanifold $\mathcal{A}_M$, and $(\mathcal{A}(M), M, u)$ the algebra of superfunctions on $\mathcal{A}_M$. For each $p \in M$ the pairing

$$\langle \cdot, \cdot \rangle_p : \mathcal{D}(\mathcal{A}_M)_p \times \mathcal{A}(M) \longrightarrow \mathbb{R}$$

is defined by

$$\langle \mu, f \rangle_p \equiv \langle F_{\alpha *}(\mu), (F_\alpha^{-1})_{U_\alpha}(f_{|U_\alpha}) \rangle_{F_\alpha^0(U_\alpha)}, \tag{47}$$

where (U_α, F_α) is a chart on $\mathcal{A}_M$ at p. By Proposition 3.4.4 the definition above is independent of the choice of a chart at p. Extending formula (47) by linearity one gets the pairing

$$\langle \cdot, \cdot \rangle_M : \mathcal{D}(\mathcal{A}_M) \times \mathcal{A}(M) \longrightarrow \mathbb{R}. \tag{48}$$

As a consequence of Propositions 3.4.4 and 3.4.5 one has the following:

Proposition 4.4.1. *Let $\mathcal{D}(\mathcal{A}_M)$ be the smooth coalgebra of a supermanifold $\mathcal{A}_M$.*
1. *For all $f, g \in \mathcal{A}(M)_0 \cup \mathcal{A}(M)_1$, $\mu \in \mathcal{D}(\mathcal{A}_M)$*

$$\langle \mu, f \cdot g \rangle_M = \sum_{(\mu)} (-1)^{|f| \cdot |\mu_{(2)}|} \langle \mu_{(1)}, f \rangle_M \langle \mu_{(2)}, g \rangle_M,$$
$$\langle \mu, 1 \rangle_M = \varepsilon_M(\mu),$$

where $\Delta_M \mu = \sum_{(\mu)} \mu_{(1)} \otimes \mu_{(2)}$.

2. *Let $F = (F^0, F_{\cdot}) : \mathcal{A}_M \to \mathcal{B}_N$ be a morphism of supermanifolds. Then*

$$\langle F_* \mu, f \rangle_N = \langle \mu, F_N f \rangle_M$$

for all $\mu \in \mathcal{D}(\mathcal{A}_M)$, $f \in \mathcal{B}(N)$.

Using Proposition A.5.6 and the above proposition one obtains the following global version of Proposition 3.4.6 [5,19].

Theorem 4.4.2. *The map*

$$\mathcal{D}(\mathcal{A}_M) \ni \omega \longrightarrow \langle \omega, . \rangle_M \in \mathcal{A}(M)^\circ$$

is an isomorphism of $\mathbb{Z}_2$*-graded coalgebras.*

Each superfunction f on a supermanifold $\mathcal{A}_M$ can be identified via the pairing (48) with a superfunction $\langle \cdot , f \rangle_M$ on $\mathcal{D}(\mathcal{A}_M)$. Assuming this identification one has:

Proposition 4.4.2. *The functor of Proposition* 4.1.6 *restricted to the subcategory* $\mathbf{SC}^<$ *of finite-dimensional smooth coalgebras*

$$\mathbf{OSC}^< \ni \mathcal{M} \longrightarrow \mathcal{M}^\wedge_M = (M, \mathcal{M}(.)^\wedge) \in \mathbf{OSM},$$
$$\mathbf{MSC}^< \ni \Phi \longrightarrow \Phi^* = (\Phi^0, \Phi^*_.) \in \mathbf{MSM}$$

is an equivalence of categories and the inverse to the functor of Theorem 4.4.1.

Acknowledgements

This work has been written during last few years with many long breaks and at many places. I would like to thank to Marjorie Batchelor, Ugo Bruzzo, Alice Rogers, Sergio Scarlatti, and Alberto Verjovsky for enlightening discussions on supergeometry. I am very grateful to Adolfo Sánchez Valenzuela for many hours of interesting discussions on supergeometry and its coalgebraic formulation. I owe special thanks to Sylvie Paycha for her enthusiasm and criticism on all stages of this project, for many discussions and comments, and for her careful critical reading of the first version of this paper.

The hospitality of the International Centre of Theoretical Physics, Trieste; Centro de Investigación en Matemáticas, Guanajuato; Institute de Recherche sur la Mathématique Avancée, Université Louis Pasteur et CRNS, Strasbourg; Laboratoire de Mathématiques Appliquees, Université Blaise Pascal, Clermont-Ferrand; and Physikalisches Institut, Universität Bonn where various parts of this work were completed is gratefully acknowledged.

Appendix A

A.1. Graded spaces

A $\mathbb{Z}_2$-graded space is a vector space V with distinguished subspaces V_0, V_1 such that V is the direct sum $V_0 \oplus V_1$ of V_0 and V_1. V_0, V_1 are called the *even* and the *odd part* of V, respectively. Similarly, an element $v \in V$ is called *even* if $v \in V_0$ and *odd* if $v \in V_1$. Any element $v \in V$ has a unique representation as a sum $v = v_0 + v_1$ of its *even* $v_0 \in V_0$ and *odd* $v_1 \in V_1$ *components*. An element $v \in V_0 \cup V_1$ is called *homogeneous*. If $v \in V_i$, $v \neq 0$ the parity $|v|$ of a homogeneous element v is defined by $|v| = i \in \mathbb{Z}_2$.

Let V, W be $\mathbb{Z}_2$-graded spaces. The space $\mathrm{Hom}(V, W)$ of all linear maps from V to W gets the natural grading

$$\mathrm{Hom}(V, W) = \mathrm{Hom}(V, W)_0 \oplus \mathrm{Hom}(V, W)_1,$$

where $f \in \mathrm{Hom}(V, W)_i$ if $f(V_j) \subset W_{i+j}$. A *morphism of $\mathbb{Z}_2$-graded spaces* is an even linear map (i.e. an element of $\mathrm{Hom}(V, W)_0$).

A *graded subspace* $W \subset V$ of a $\mathbb{Z}_2$-graded space V is a vector subspace of V with the $\mathbb{Z}_2$-grading given by $W = (W \cap V_0) \oplus (W \cap V_1)$.

The *direct sum* $V \oplus W$ *of $\mathbb{Z}_2$-graded spaces* V, W is the direct sum $V \oplus W$ of vector spaces V, W with the $\mathbb{Z}_2$-grading

$$(V \oplus W)_0 = V_0 \oplus W_0, \qquad (V \oplus W)_1 = V_1 \oplus W_1.$$

The *tensor product* $V \otimes W$ *of $\mathbb{Z}_2$-graded spaces* V, W is the tensor product $V \otimes W$ of vector spaces with the $\mathbb{Z}_2$-grading

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j.$$

Let V, W be $\mathbb{Z}_2$-graded spaces. The *twisting morphism* $T : V \otimes W \longrightarrow W \otimes V$ is a morphisms of graded spaces defined by

$$T(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

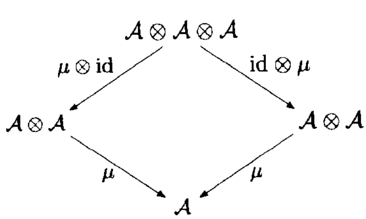
for all $v \in V_0 \cup V_1$, $w \in W_0 \cup W_1$.

A.2. Graded algebras

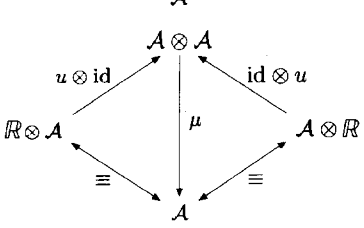
Definition A.2.1. A triple $(\mathcal{A}, \mu, u)$ where $\mathcal{A}$ is a $\mathbb{Z}_2$-graded space and μ, u are morphisms of $\mathbb{Z}_2$-graded spaces

$$\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \quad \text{(multiplication)}$$
$$u : \mathbb{R} \longrightarrow \mathcal{A} \quad \text{(unit)}$$

is called a $\mathbb{Z}_2$-graded algebra if the diagrams



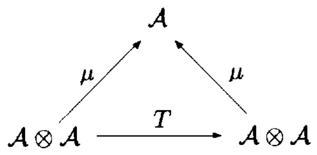
(associativity)



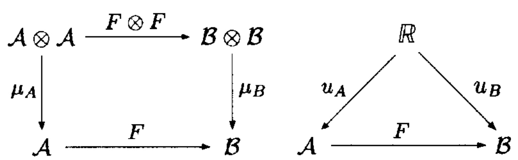
(unitarity)

commute.

A $\mathbb{Z}_2$-graded algebra A is called *commutative* if the following diagram commutes:



A *morphism of* $\mathbb{Z}_2$*-graded algebras* $F : A \to B$ is a morphism of graded spaces such that the diagrams



commute.

The *tensor product of* $\mathbb{Z}_2$*-graded algebras* (A, μ_A, u_A), (B, μ_B, u_B) is the tensor product $A \otimes B$ of $\mathbb{Z}_2$-graded spaces with the $\mathbb{Z}_2$-graded algebra structure given by

$$\mu_{A\otimes B} : A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes T \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B,$$

$$u_{A\otimes B} : \mathbb{R} \equiv \mathbb{R} \otimes \mathbb{R} \xrightarrow{v_A \otimes u_B} A \otimes B.$$

Definition A.2.2. A bigraded algebra A is a $\mathbb{Z}_2$-graded algebra with a $\mathbb{Z}_+$-grading $A = \bigoplus_{i=1}^{\infty} A^i$ such that:

1. for energy $i \in \mathbb{Z}_+$, A^i is a $\mathbb{Z}_2$-graded subspace of A;
2. $u(\mathbb{R}) \subset A^0$;
3. for every $i, j \in \mathbb{Z}_+$, $\mu(A^i \otimes A^j) \subset A^{i+j}$.

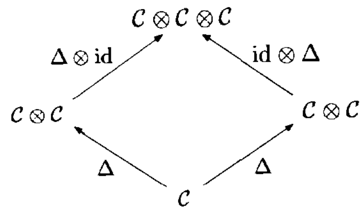
A.3. Graded coalgebras

Definition A.3.1. A triple (C, Δ, ε) where C is a $\mathbb{Z}_2$-graded space and Δ, ε are morphisms of $\mathbb{Z}_2$-graded spaces

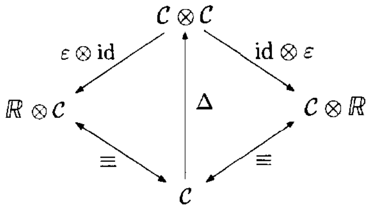
$$\Delta : C \longrightarrow C \otimes C \quad \text{(comultiplication)}$$

$$\varepsilon : C \longrightarrow \mathbb{R} \quad \text{(counit)}$$

is called a $\mathbb{Z}_2$-graded coalgebra if the diagrams



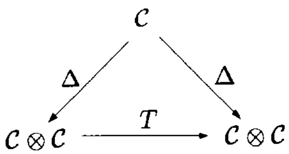
(associativity)



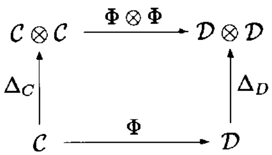
(unitarity)

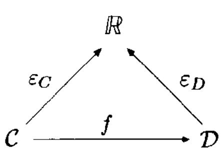
commute.

A $\mathbb{Z}_2$-graded coalgebra $\mathcal{C}$ is called *cocommutative* if the following diagram commutes:



A *morphism of $\mathbb{Z}_2$-graded coalgebras* $\Phi : \mathcal{D} \longrightarrow \mathcal{D}$ is a morphism of graded spaces such that the diagrams





commute.

The *tensor product of $\mathbb{Z}_2$-graded coalgebras* $(\mathcal{C}, \Delta_c, \varepsilon_C)$, $(\mathcal{D}, \Delta_D, \varepsilon_D)$ is the tensor product $\mathcal{C} \otimes \mathcal{D}$ of $\mathbb{Z}_2$-graded spaces with the $\mathbb{Z}_2$-graded coalgebra structure given by

$$\Delta_{C\otimes D} : C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}_C \otimes T \otimes \mathrm{id}_D} C \otimes D \otimes C \otimes D,$$

$$\varepsilon_{C\otimes D} : C \otimes D \xrightarrow{\varepsilon_C \otimes \varepsilon_D} \mathbb{R} \otimes \mathbb{R} \equiv \mathbb{R}.$$

Let $\mathcal{C}$ be a $\mathbb{Z}_2$-graded coalgebra. For any $c \in \mathcal{C}$, Δc can be written as a (nonunique) sum of simple tensors. It is convenient to use the following so called *sigma notation*:

$$\Delta c = \sum_{(c)} c_{(1)} \otimes c_{(2)},$$

where one can assume that all $c_{(1)}, c_{(2)}$ are homogeneous. Similarly, by the coassociativity one can write

$$\Delta^k c = (\Delta \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{k-1}) \circ \cdots \circ \Delta \otimes \mathrm{id} \circ \Delta$$
$$= \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(k+1)}.$$

A $\mathbb{Z}_2$-graded coalgebra $\mathcal{C}$ is *irreducible* if any of two nonzero $\mathbb{Z}_2$-graded subcoalgebras have nonzero intersection. A maximal irreducible $\mathbb{Z}_2$-graded subcoalgebra of $\mathcal{C}$ is called an *irreducible component of* $\mathcal{C}$. A $\mathbb{Z}_2$-graded coalgebra is *simple* if it has no nonzero proper $\mathbb{Z}_2$-graded subcoalgebras. A $\mathbb{Z}_2$-graded coalgebra is *pointed* if all simple $\mathbb{Z}_2$-graded subcoalgebras are one-dimensional.

The structure theorem for cocomutative coalgebras [40] is also valid in the $\mathbb{Z}_2$-graded case [5,19].

Theorem A.3.1. *Any cocomutative $\mathbb{Z}_2$-graded coalgebra is a direct sum of its irreducible components.*

An element g of a $\mathbb{Z}_2$-graded coalgebra is called *group-like* if $\Delta g = g \otimes g$. We denote the set of all group-like elements of a $\mathbb{Z}_2$-graded coalgebra $\mathcal{C}$ by $G(\mathcal{C})$. For all $g \in G(\mathcal{C})$, $\mathbb{R}g \subset \mathcal{C}_0$ is a one-dimensional simple $\mathbb{Z}_2$-graded subcoalgebra of $\mathcal{C}$. When a $\mathbb{Z}_2$-graded cocomutative coalgebra is pointed, each irreducible component $\mathcal{C}_g$ of $\mathcal{C}$ is uniquely determined by a unique group-like element g contained in $\mathcal{C}_g$. Then the direct sum decomposition takes the form

$$\mathcal{C} = \bigoplus_{g \in G(\mathcal{C})} \mathcal{C}_g.$$

In the case of a pointed irreducible coalgebra $\mathcal{C}$ some further structure information is encoded in so called coradical filtration [40]. We shall briefly describe the $\mathbb{Z}_2$-graded version of this construction and theorem [5,19].

Definition A.3.2. A filtration of a $\mathbb{Z}_2$-graded coalgebra $\mathcal{C}$ is a family $\{\mathcal{C}^{(k)}\}_{k=0}^{0}$ of $\mathbb{Z}_2$-graded subcoalgebras such that
1. For any $k \leq k'$, $\mathcal{C}^{(k)}$ is a $\mathbb{Z}_2$-graded subcoalgebra of $\mathcal{C}^{(k')}$.
2. $\mathcal{C} = \bigcup_{k \geq 0} \mathcal{C}^{(k)}$.
3. $\Delta \mathcal{C}^{(k)} = \sum_{i=0}^{k} \mathcal{C}^{(k-i)} \otimes \mathcal{C}^{(i)}$, for all $k \geq 0$.

Let $\mathcal{C}_g$ be a pointed irreducible $\mathbb{Z}_2$-graded coalgebra and g its unique group-like element. There is direct sum decomposition

$$\mathcal{C}_g = \mathbb{R}g \oplus \mathcal{C}_g^+,$$

where $\mathcal{C}_g^+ = \ker \varepsilon_{\mathcal{C}}$. Let $\pi_g^+ : \mathcal{C}_g \longrightarrow \mathcal{C}_g^+$ be the projection on the second factor. We define family $\{\mathcal{C}_g^{(k)}\}_{k=0}^{\infty}$ of $\mathbb{Z}_2$-graded subcoalgebras of $\mathcal{C}_g$:

$$C_g^{(0)} = \mathbb{R}g,$$

$$C_g^{(k)} = \ker\left(\bigotimes^{k+1} \pi_g^+\right) \circ \Delta^k, \quad k \geq 1.$$

Since both π_g^+ and Δ^k are morphisms of $\mathbb{Z}_2$-graded spaces, $C_g^{(k)}$ is a $\mathbb{Z}_2$-graded subspace of C_g for all $k \geq 1$.

Proposition A.3.1. *If $c \in C_g^{(k)+} = C_g^{(k)} \cap C_g^+$, then*

$$\Delta c = g \otimes c^1 + c^1 \otimes g + y,$$

where

$$y \in \sum_{i=1}^{k-1} C_g^{(i)+} \otimes C_g^{(k-i)+}.$$

Theorem A.3.2. *The family $\{C_g^{(k)}\}_{k=0}^{\infty}$ is a filtration of the coalgebra C_g.*

The filtration of the theorem above is called the *coradical filtration*.

Let g be a group-like element of a $\mathbb{Z}_2$-graded coalgebra C. An element $p \in C$ is called *primitive with respect to g* if

$$\Delta p = p \otimes g + g \otimes p.$$

We denote the set of all elements $p \in C$ primitive with respect to g by $P_g(C)$. Note that $P_g(C) \subset C_g$, where C_g is the irreducible component containing g. By Proposition A.3.1

$$C_g^{(1)} = \mathbb{R}g \oplus P_g(C).$$

In particular $P_g(C)$ is a $\mathbb{Z}_2$-graded subspace of C. In case of a pointed irreducible $\mathbb{Z}_2$-graded coalgebra C we denote by $P(C)$ the space of all primitive elements with respect to a unique group-like element in C.

Proposition A.3.2. *Let $C, \mathcal{D}$ be $\mathbb{Z}_2$-graded cocommutative coalgebras and Φ, Ψ morphisms of $\mathbb{Z}_2$-graded coalgebras $\mathcal{D} \to C$. Suppose C is pointed irreducible, then $f = g$ if and only if* $\mathrm{Im}(\Phi - \Psi) \cap P(C) = \{\mathrm{o}\}$.

Definition A.3.3. A bigraded coalgebra is a $\mathbb{Z}_2$-graded coalgebra C with a $\mathbb{Z}_+$-grading $C = \bigoplus_{k\geq 0} C^k$ such that:
1. For every $k \geq 0$, C^k is a $\mathbb{Z}_2$-graded subspace of C.
2. $\varepsilon(C^k) = 0$ for all $k \geq 1$.
3. For every $k \geq 0$, $\Delta(C^k) \subset \bigoplus_{i=0}^{k} C^i \otimes C^{k-i}$.

A bigraded coalgebra C is called strictly bigraded if $C^0 = \mathbb{R}$ and C^1 coincides with the space $P(C)$ of all primitive elements of C.

Note that the condition $\mathcal{C}^0 = \mathbb{R}$ implies that a strictly bigraded coalgebra is pointed irreducible. The relation between the $\mathbb{Z}_+$-grading and the coradical filtration in strictly bigraded coalgebras is given by the following:

Proposition A.3.3. *A bigraded coalgebra $\mathcal{C}$ with $\mathcal{C}^0 = \mathbb{R}$ is strictly bigraded if and only if for every $k \geq 0$*

$$\mathcal{C}^{(k)} = \bigoplus_{i=0}^{k} \mathcal{C}^i .$$

A.4. Graded bialgebras

Definition A.4.1. A system $(\mathcal{B}, \mu, u, \Delta, \varepsilon)$ where $\mathcal{B}$ is a $\mathbb{Z}_2$-graded space and $\mu, u, \Delta, \varepsilon$ are morphisms of $\mathbb{Z}_2$-graded spaces

$$\begin{array}{ll} \mu : \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B} & \Delta : \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B} \\ u : \mathbb{R} \longrightarrow \mathcal{B} & \varepsilon : \mathcal{B} \longrightarrow \mathbb{R} \end{array}$$

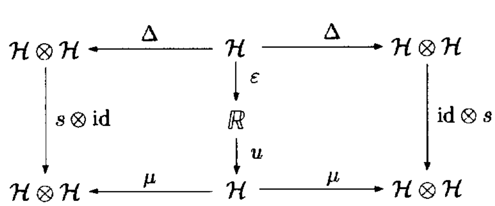
is called a $\mathbb{Z}_2$-graded bialgebra if
1. $(\mathcal{B}, \mu, u)$ is a $\mathbb{Z}_2$-graded algebra.
2. $(\mathcal{B}, \Delta, \varepsilon)$ is a $\mathbb{Z}_2$-graded coalgebra.
3. Δ and ε are morphisms of $\mathbb{Z}_2$-graded algebras.

Note that condition 3 can be replaced by requirement that μ and u are morphisms of $\mathbb{Z}_2$-graded coalgebras.

Definition A.4.2. A bigraded bialgebra is a $\mathbb{Z}_2$-graded bialgebra which is both a bigraded algebra and bigraded coalgebra with respect to the same $\mathbb{Z}_+$-grading.

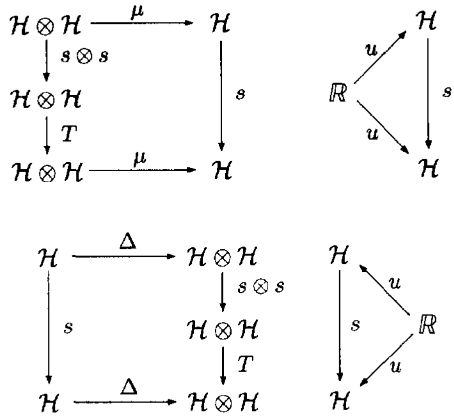
A bigraded bialgebra is called cocommutative, pointed, irreducible, strictly bigraded, if it is so with respect to its coalgebra structure.

Definition A.4.3. A $\mathbb{Z}_2$-graded bialgebra $\mathcal{H}$ is called a $\mathbb{Z}_2$-graded Hopf algebra if there exists a morphism of $\mathbb{Z}_2$-graded spaces $s : \mathcal{H} \to \mathcal{H}$ such that the diagram



commutes. The morphism is called the antipode of $\mathcal{H}$.

The antipode if exists is unique. One can also show that s is a $\mathbb{Z}_2$-graded algebra and coalgebra antimorphism, i.e. the following diagrams are commutative.



If $\mathcal{H}$ is a bigraded bialgebra and the antipode exists, it necessarily respects $\mathbb{Z}_+$-grading.

A.5. Dual coalgebras

Let (A, M, u) be an algebra over $\mathbb{R}$ with multiplication $M : A \otimes A \to A$ and unit $u : \mathbb{R} \to A$. We denote by A° the subspace of the full algebraic dual A' consisting of all elements $\alpha \in A'$ such that $\ker \alpha$ contains a cofinite ideal of A.

Proposition A.5.1. *Let A, B be $\mathbb{Z}_2$-graded algebras and $F : A \to B$ a morphism of $\mathbb{Z}_2$-graded algebras. Then:*
1. *A° is a linear subspace of A'.*
2. *Let $F' : B' \to A'$ be dual to $F : A \to B$. Then $F'(B^\circ) \subset A^\circ$.*
3. *$A^\circ \otimes B^\circ = (A \otimes B)^\circ$.*
4. *Let $M' : A \to (A \otimes A)'$ be the dual to the multiplication in A. Then $M'(A^\circ) \subset A^\circ \otimes A^\circ$.*

Proposition A.5.2. *Let (A, M, u) be a $\mathbb{Z}_2$-graded algebra over $\mathbb{R}$. Then the maps*

$$\Delta \equiv M'_{|A^\circ} : A^\circ \longrightarrow A^\circ \otimes A^\circ,$$
$$\varepsilon \equiv u'_{|A^\circ} : A^\circ \longrightarrow \mathbb{R},$$

define on A° a structure of $\mathbb{Z}_2$-graded coalgebra. If A is $\mathbb{Z}_2$-graded commutative then A° is $\mathbb{Z}_2$-graded cocommutative.

Definition A.5.1. The coalgebra $(A^\circ, \Delta, \varepsilon)$ of Proposition A.5.2 is called the dual coalgebra of (A, M, u).

The following property of $(A^\circ, \Delta, \varepsilon)$ may serve as independent definition of dual coalgebra.

Proposition A.5.3. *A° is the maximal coalgebra in A', i.e. if for $\alpha \in A'$, $M'(\alpha) \in A' \otimes A'$ then $\alpha \in A^\circ$.*

The dual coalgebras of $\mathbb{Z}_2$-graded algebras of superfunctions of BLK supermanifolds have been analysed in [5,19]. Here we briefly present structure results used in the main text.

Proposition A.5.4. *Let $\mathcal{A}(M)$ be the $\mathbb{Z}_2$-graded algebra of superfunctions on a supermanifold $\mathcal{A}_M$. Then*
1. *$\mathcal{A}(M)^\circ$ is a pointed $\mathbb{Z}_2$-graded cocomutative coalgebra.*
2. *Each group-like element of $\mathcal{A}(M)^\circ$ is of the form*

$$\delta_p = \langle p, \cdot\rangle : \mathcal{A}(M) \ni f \longrightarrow f^0(p) \in \mathbb{R}$$

for some $p \in M$.

By the structure theorem for pointed $\mathbb{Z}_2$-graded cocomutative coalgebras (Theorem A.3.1), one has the following:

Proposition A.5.5. *$\mathcal{A}(M)^\circ$ is the direct sum of pointed irreducible coalgebras*

$$\mathcal{A}(M)^\circ = \bigoplus_{p\in M} \mathcal{A}(M)^\circ_p,$$

where $\mathcal{A}(M)^\circ_p$ denotes the irreducible component containing the group-like element δ_p.

Applying the structure theorem for pointed irreducible $\mathbb{Z}_2$-graded coalgebras (Theorem A.3.2) one gets for each irreducible component $\mathcal{A}(M)^\circ_p$ the coradical filtration

$$\mathcal{A}(M)^\circ_p = \bigcup_{p\in M} \mathcal{A}(M)^{\circ(k)}_p,$$

with

$$\mathcal{A}(M)^{\circ(0)}_p = \mathbb{R}p,$$
$$\mathcal{A}(M)^{\circ(1)}_p = \mathbb{R}p \oplus P(\mathcal{A}(M)^\circ_p),$$

where $P(\mathcal{A}(M)^\circ_p)$ is the space of all primitive with respect to δ_p elements of $\mathcal{A}(M)^\circ$. A more detailed description is given by the following [5,19]:

Proposition A.5.6. *Let $\mathcal{A}(M)^\circ_p = \bigcup_{k\geq 0} \mathcal{A}(M)^{\circ(k)}_p$ be the coradical filtration of the irreducible component $\mathcal{A}(M)^\circ_p$ of the dual coalgebra $\mathcal{A}(M)^\circ$. Then for each $k \geq 0$*

$$\mathcal{A}(M)^{\circ(k)}_p = \{\alpha \in \mathcal{A}(M)^{\circ(k)} : \langle\alpha, I_p^{k+1}\rangle = 0\} = (\mathcal{A}(M)/I_p^{k+1})',$$

where I_p is a maximal ideal in $\mathcal{A}(M)$ consisting of all superfunctions $f \in \mathcal{A}(M)$ such that $f^0(p) = 0$.

References

[1] A.M. Baranov, Yu.I. Manin, I.V. Frolov, A.S. Schwarz, A superanalog of the Selberg trace formula and multiloop contributions for fermionic strings, Commun. Math. Phys. 111 (1987) 373–392.

[2] C. Bartocci, U. Bruzzo, Some remarks on the differential-geometric approach to supermanifolds, J. Geom. Phys. 4 (1987) 391–404.
[3] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, The Geometry of Supermanifolds, Kluwer Academic Publications, Dordrecht, 1991.
[4] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, V.G. Pestov, Foundations of supermanifold theory: the axiomatic approach, Diff. Geom. Appl. 3 (1993) 135–155.
[5] B. Batchelor, In search of the graded manifold of maps between graded manifolds, in: Complex Differential Geometry and Supermanifolds in Strings and Fields, Lecture Notes in Physics, vol. 311, Springer, Berlin, 1988, pp. 62–113.
[6] M. Batchelor, P. Bryant, Graded Riemann surfaces, Commun. Math. Phys. 114 (1988) 243–255.
[7] F.A. Berezin, A.A. Kirillov, D. Leites (eds.), Inotroduction to Superanalysis, Mathematical Physics and Applied Mathematics, vol. 9, Reidel, Dordrecht, 1987.
[8] F.A. Berezin, D Leites, Supermanifolds, Soviet Math. Dokl. 16 (1975) 1218–1222.
[9] C.P. Boyer, S. Gitler, The theory of $\mathcal{G}^\infty$ supermanifolds. Trans. Amer. Math. Soc. 285 (1984) 241–267.
[10] J. Dell, L. Smolin, Graded manifold theory as the geometry of superspace, Commun. Math. Phys. 66 (1979) 335.
[11] B.S. De Witt, Supermanifolds, Cambridge University Press, London, 1984.
[12] K. Gawedzki, Supersymmetries – Mathematics of supergeometry, Ann. Inst. H. Poincaré Sect. A (N.S.) 27 (1977) 335–366.
[13] R. Giachetti, R. Ragionieri, R. Ricci, Symplectic structures on graded manifolds, J. Diff. Geom. 16 (1981) 247–253.
[14] S.B. Giddings, P. Nelson, The geometry of super Riemann surfaces, Commun. Math. Phys. 116 (1988) 607–634.
[15] P. Green, On holomorphic graded manifolds, Proc. Amer. Math. Soc. 85 (1982) 587–590.
[16] R.S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. 7 (1982) 65–222.
[17] D. Hernández Ruipérez, J. Muñoz Masqué, Global variational calculus on graded manifolds I, J. Math. Pure Appl. 63 (1984) 283–309; Global variational calculus on graded manifolds II, J. Math. Pure Appl. 64 (1985) 87–104.
[18] A. Jadczyk, K. Pilch, Superspaces and supersymmetries, Commun. Math. Phys. 78 (1981) 373–390.
[19] B. Kostant, Graded manifolds, graded Lie theory and prequantization, in: Differential Geometric methods in Mathematical Physics, Lecture Notes in Mathematics, vol. 570, Springer, Berlin, 1977, pp. 177–306.
[20] S. Lang, Introduction to Differentiable Manifolds, Interscience, New York, 1962.
[21] C. LeBrun, M. Rothstein, Moduli of super Riemann surfaces, Commun. Math. Phys. 117 (1988) 159–176.
[22] D. Leites, Introduction to the theory of supermanifolds, Russ. Math. Surv. 35 (1) (1980) 1–64.
[23] J. Lott, Supersymmetric path integral, Commun. Math. Phys. 108 (1990) 605–629.
[24] J. Lott, Torsion constraints in supergeometry, Commun. Math. Phys. 133 (1990) 563–615.
[25] Y. Manin, Gauge Field Theory and Complex Geometry, Springer, Berlin, 1988.
[26] V. Molotkov, Infinite-dimensional $\mathbb{Z}_2^k$-supermanifolds, ICTP Preprint IC/84/183 (1984); Banach supermanifolds, in: H.D. Doebner, T.D. Palev (Eds.), Differential Geometric Methods in Theoretical Physics, Proceedings of the 13th International Conference on Differential Geometric Methods in Theoretical Physics, Shumen, Bulgaria, 20–24 August, 1984, World Scientific, Singapore, 1986, pp. 117–125; Sheaves of authomorphisms and invariants of Banach supermanifolds, in: Mathematics and Education in Mathematics, Proceedings of the 15th Spring Conference of the Union of Bulgarian Mathematicians, Sunny Beach, 2–6 April 1986, Sofia, BAN, 1986, pp. 271–283.
[27] P. Nelson Introduction to supermanifolds, Int. J. Mod. Phys. A 3 (1988) 587–590.
[28] R.S. Palais, Foundations of Global Non-linear Analysis, Benjamin Company, New York, 1968.
[29] I.B. Penkov, $\mathcal{D}$-Modules on supermanifolds, Invent. Math. 71 (1983) 501–512.
[30] J.M. Rabin, L. Crane, Global properties of supermanifolds, Commun. Math. Phys. 100 (1985) 141–160.
[31] A. Rogers, A global theory of supermanifolds, J. Math. Phys. 21 (1980) 1352–1365.
[32] M. Rothstein, Deformations of complex supermanifolds, Proc. Amer. Math. Soc. 95 (1985) 255–260.
[33] M. Rothstein, The axioms of supermanifolds and a new structure arising from them, Trans. Amer. Math. Soc. 297 (1986) 159–180.

[34] M. Rothstein, The structure of supersymplectic supermanifolds, in: C. Bartocci, U. Bruzzo, R. Cianci (Eds.), Differential Geometric Methods in Mathematical Physics, Lecture Notes in Physics, vol. 375, Springer, Berlin, 1991, pp. 331–343.
[35] T. Schmitt, Superdifferential geometry, Report 05/84 des Imath, Berlin, 1984.
[36] T. Schmitt, Infinite-dimensional supergeometry, and on supergeometry and its application to physics, in: Seminar Analysis of the Karl-Weierstrass-Institute 1986/87 Teubner-Texte zur Mathematik, Leipzig, 1987.
[37] T. Schmitt, Infinitedimensional Supermanifolds I, Report 08/88 des Karl-Weierstrass-Instituts für Mathematik, Berlin, 1988.
[38] T. Schmitt, Infinitedimensional supermanifolds II, III, Mathematica Gottingensis, Schriftenreihe des SFBs Geometrie und Analysis, Heft 33, 34 (1990), Götingen, 1990.
[39] A.A. Schwarz, Supergravity, complex geometry and G-structures, Commun. Math. Phys. 87 (1982) 37–63; To the definition of superspace, Theor. Math. Phys. 60 (1984) 655–659.
[40] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[41] A.A. Voronov, Mapping of supermanifolds, Theor. Math. Phys. 60 (1984) 660–664.